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Thermal convection in a horizontal porous layer affected by rotation

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of science in Applied Mathematics

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SUMMARY

Thermal convection in a horizontal porous layer affected by rotation

This thesis examines the thermal convection of an infinite horizontal layer occupied by a porous medium permeated by an incompressible viscous fluid and affected by rotation. The problem is discussed assuming that the porous medium is controlled by Brinkman model. Both stationary and overstability cases were discussed. It has been shown that when the fluid is heated from above, no instabilities can occurs. However when the fluid is heated from below we obtain stationary instability and overstability. Analytical results are obtained when both boundaries are free. Numerical results were also obtained when both boundaries are free or rigid using Chebyshev Tau method. The effects of rotation and permeability of porous medium were discussed.

Contents

Chapter 1

CHAPTER ONE

Introduction:

In recent years, considerable attention has been paid to thermal instability theory in fluid. The convection in a thin horizontal layer of fluid heated from below is well suited to illustrate the many facts, mathematical and physical, of the general theory of hydrodynamic stability. The earliest experiments demonstrated the onset of thermal instability in fluids, are those of Benard [9,10], who carried out his experiments on very thin layers of an incompressible viscous fluid, standing on a leveled metallic plate maintained at a constant temperature. The upper surface was usually free and being in contact with the air was at a lower temperature. He found that a certain critical adverse temperature gradient must be exceeded before instability can sets in.

Rayleigh [43] provided a theoretical basis for Benard's experimental results. He showed that the numerical value of the non-dimensional parameter, (known as Rayleigh number)

$$R = \frac{d^4 g\alpha\beta}{\nu\,\kappa}$$

decides whether a layer of fluid heated from below is stable or not, where g is the acceleration of gravity, β is the uniform adverse temperature gradient, d is the depth of the layer, α, ν and κ are coefficients of volume expansion, kinematic viscosity and thermal diffusivity respectively. Jeffreyes [29,30], Low [35] and Pellew and Southwell [40] extended Rayleigh's analysis using different boundary conditions.

The instability of a layer of fluid heated from below and subjected to Coriolis forces have been studied by Chandrasekher [13] and Chandrasekher and Elbert [14] for stationary convection and overstability respectively. They showed that the presence of Coriolis forces usually inhibits the onset of thermal convection and the instability of the fluid layer depends on the value of the non-dimensional number, (known as Taylor number)

$$T = 4\frac{d^4\Omega^2}{\nu^2}$$

where Ω is the magnitude of the angular velocity.

The stability of Benard problem for a fluid in a porous medium has been examined by Horton and Rogers [28], Lapwood [34], Wooding [47], and Elder [17] using Darcy's law which states that the hydraulic pressure gradient is proportional to the fluid velocity and to its viscosity, and is inversely proportional to the permeability. The resistance term, calculated from Darcy's law, replaces the Navier-stokes viscous term. A modification of Darcy's law has been suggested by Brinkman [11,12]. His model assumes that the viscosity term in Navier-Stokes equation should be included in the equation of motion together with the resistance term.

There has been considerable interest in the study of different problems in the presence of porous medium by several authors. Yamamoto and Iwamura [48] studied the flow with convective acceleration through a porous medium using Brinkman model. Rudraiah, et al. [44] have examined the effects of nonuniform thermal gradient and adiabatic boundaries on convection in porous media for free, rigid and mixed boundaries using Brinkman model. Georgiadis and Catton [21] studied the Prandtl number effect on Benard convection in porous medium. Kladias and Prasad [32] studied the thermoconvection instabilities in a horizontal porous layer heated from below when the boundaries are free. Pradeep and Sri Krishna [41] studied the Rayleigh-Benard convection problem in a Boussinesquian, viscoelastic fluid-filled with high-porosity medium numerically using the single-term Galerkin technique for free-free, free-rigid and rigid-rigid boundaries. Allihyani [8] studied the Benard convection in a horizontal porous layer permeated by a conducting fluid in the presence of magnetic field and Coriolis force. Hatim [26] studied the thermosolutal convection in a horizontal porous layer affected by a magnetic field. Hill [27] studied convection induced by the selective absorption of radiation for the Brinkman model. Kasmi [31] studied the thermosolutal convection in a horizontal porous layer affected by a magnetic field and rotation. Ramambason and Vasseur [42] studied the influence of a magnetic field on natural convection in a shallow porous enclosure saturated with a binary fluid. Gaikwad et al. [20] studied analytically the double diffusive in a horizontal anisotropic porous layer

saturated with a Boussinesq fluid. An extensive review articles on convection in porous medium can be found in Nield and Bejan [37].

Rotating Rayleigh-Benard convection has important applications in geophysical and astrophysical flows as well as industrial applications. Many flows in nature are driven by buoyant convection and subsequently modulated by rotation. One of these systems is relevant to numerous astrophysical and geophysical phenomena, including convection in arctic ocean, in the earth's outer core, in the interior of gaseous giant planets and the outer layer of the sun. Thus the problem is of interest in a wide range of sciences including geology, oceanography, climatology and astrophysics.

In this thesis thermal convective instability in a horizontal porous layer using Brinkman model is studied for both stationary and overstability cases with different boundary conditions. Analytical and numerical solutions were obtained. The numerical method used to solve the problem is the Chebyshev Tau method. This method is better suited to the solution of hydrodynamic stability problems than expansions in other sets of orthogonal polynomials. It was first introduced by Lanczos [33] and Clenshaw [16] to obtain numerical solutions of linear differential equations. Fox [18], Fox and Parker [19] and Orszag [38,39] used Chebyshev polynomials to solve the Orr-Sommerfeld equation. Their computational treatment of this equation using Chebyshev polynomials established the prominence and viability of this method. Several authors used this method to obtain numerical solutions of thermal stability problems (Abdullah and Lindsay [4-5], Abdullah [1,2], Hassanien et al. [24,25] , Abdullah [3], Straughan [45,46], Al-idrous and Abdullah [7], Gheorghiu [22], and Gheorghiu and Dragomirescu [23]).

In chapter two the Chebyshev Tau method is described in detail. In chapter three the thermal convective instability in a horizontal porous layer affected by rotation is studied. The stability is considered for the cases of stationary and overstability. The eigenvalue problem is solved together with the boundary conditions numerically for the case when the fluid layer is heated from below using the method of expansion of Chebyshev polynomials.

Chapter 2

CHAPTER TWO

Chebyshev Tau method for boundary value problems

The expansions of Chebyshev polynomials have been used to obtain solutions of many hydrodynamic stability problems. These types of polynomials are appropriate to solve hydrodynamic stability problems than other sets of orthogonal polynomials and results of high accuracy are obtained using Chebyshev approximations. In this chapter we shall present some properties of these polynomials together with an example of using this method to solve a hydrodynamic stability problem.

2.1 Some properties of Chebyshev polynomials:

The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined in the interval $[-1,1]$ by the relation

$$T_n(\cos\theta) = \cos(n\theta) \ , \qquad x = \cos\theta. \tag{2.1.1}$$

If the range of the variable x is the interval $[-1,1]$, then the range of the corresponding variable θ can be taken as $[0,\pi]$. These ranges are traversed in opposite directions, since $x = -1$ corresponds to $\theta = \pi$ and $x = 1$ corresponds to $\theta = 0$. Thus

$$|T_n(x)| \le 1, \qquad |x| \le 1\,. \tag{2.1.2}$$

We may immediately deduce from (2.1.1) that the first few Chebyshev polynomials are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \ldots$$

From the trigonometric identity

$$\cos((n+m)\theta) + \cos((n-m)\theta) = 2\cos(n\theta)\cos(m\theta)$$

we deduce the recurrence relation (see Mason and Handscomb [36])

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n \geq 2. \tag{2.1.3}$$

In terms of the variable x, we may write (2.1.1) as

$$T_n(x) = \cos(n\cos^{-1}(x)) \tag{2.1.4}$$

from which we can show that

$$T_n(-1) = \cos(n\cos^{-1}(-1)) = \cos(n\pi) = (-1)^n$$

$$T_n(1) = \cos(n\cos^{-1}(1)) = \cos(n0) = 1$$

i.e. $$T_n(\pm 1) = (\pm 1)^n, \qquad n \geq 0. \tag{2.1.5}$$

By differentiating (2.1.1) with respect to θ, we find that

$$T_n'(\cos\theta) = -\sin(n\cos^{-1}x)\frac{(-n)}{\sqrt{1-x^2}}$$

$$= \frac{n\sin(n\theta)}{\sin\theta}, \qquad n \geq 0. \tag{2.1.6}$$

Evaluation of this expression at $x = \pm 1$, we obtain

$$T_n'(1) = n^2, \qquad n \geq 0. \tag{2.1.7}$$

and

$$T_n'(-1) = (-1)^{n-1}\, n^2, \quad n \geq 0. \tag{2.1.8}$$

The Chebyshev polynomials are orthogonal over $[-1,1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{(1-x^2)}}$, i.e.

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{(1-x^2)}}\, dx = \begin{cases} \pi & ; \quad n = m = 0 \\ \frac{\pi}{2} & ; \quad n = m \neq 0 \\ 0 & ; \quad n \neq m \end{cases}$$

where n, m are positive integers. This can be written as

$$\int_{-1}^{1} T_n(x)T_m(x)w(x)dx = C_n\delta_{mn} \tag{2.1.9}$$

where

$$C_n = \frac{\pi}{2}\begin{cases} 2 & ; \quad n = 0 \\ 1 & ; \quad n > 0 \end{cases}$$

and δ_{mn} is the Kronecher delta defined by

$$\delta_{mn} = \begin{cases} 1 & ; \quad n = m \\ 0 & ; \quad n \neq m. \end{cases}$$

For $f(x)$ defined in the interval $[-1,1]$ and derfferentiable, the Chebyshev expansion is given by (see Fox [18])

$$f(x) = \sum_{n=0}^{\infty} a_{n+1}T_n(x) \tag{2.1.10}$$

where

$$a_{n+1} = \frac{1}{C_n}\int_{-1}^{1} f(x)T_n(x)w(x)dx\,, \qquad n \geq 0.$$

The derivative of the function $f(x)$ expanded in Chebyshev polynomials can be represented formally as

$$f'(x) = \sum_{n=0}^{\infty} a_{n+1}T_n'(x), \tag{2.1.11}$$

where $T_n'(x)$ is a polynomial of degree $(n-1)$. We can write $T_n'(x)$ as an expansion of Chebyshev polynomials (see Abdullah and Lindsay [6]) i.e.

$$T_n'(x) = \sum_{m=0}^{\infty} B_{m+1,n+1} T_m(x), \qquad n \geq 0. \qquad (2.1.12)$$

where $\underline{B}$ is the derivative matrix and where

$$B_{m+1,n+1} = 0 \qquad if \qquad m \geq n.$$

From (2.1.12) we have

$$\int_{-1}^{1} T_n'(x) T_r(x) w(x) dx = \sum_{m=0}^{\infty} B_{m+1,n+1} \int_{-1}^{1} T_r(x)\, T_m(x) w(x) dx.$$

Using (2.1.9) ,we obtain

$$\int_{-1}^{1} T_n'(x) T_r(x) w(x) dx = \sum_{m=0}^{\infty} B_{m+1,n+1} \begin{cases} \pi \delta_{rm} \ , & r = 0, \\ \frac{\pi}{2} \delta_{rm} \ , & r > 0. \end{cases} \qquad (2.1.13)$$

Now when $r = 0$ then equation (2.1.13) becomes

$$\pi\, B_{1,n+1} = \int_{-1}^{1} T_n'(x) w(x) dx$$

i. e.

$$B_{1,n+1} = \frac{1}{\pi} \int_{-1}^{1} T_n'(x) w(x) dx, \qquad (2.1.14)$$

and when $r \geq 1$ we have

$$\frac{\pi}{2}\, B_{r+1,n+1} = \int_{-1}^{1} T_n'(x) T_r(x) w(x) dx,$$

i. e. $$B_{r+1,n+1} = \frac{2}{\pi}\int_{-1}^{1} T_n'(x)T_r(x)w(x)dx\ . \qquad (2.1.15)$$

Using (2.1.6) equation (2.1.14) becomes

$$B_{1,n+1} = \frac{n}{\pi}\int_{0}^{\pi} \frac{\sin(n\theta)}{\sin\theta}d\theta. \qquad (2.1.16)$$

Similarly equation (2.1.15) becomes

$$B_{r+1,n+1} = \frac{n}{\pi}\int_{0}^{\pi} \frac{2\sin(n\theta)\cos(r\theta)}{\sin\theta}d\theta.$$

Applying the identity $2\sin(n\theta)\cos(r\theta) = \sin(n+r)\theta + \sin(n-r)\theta$, then

$$B_{r+1,n+1} = \frac{n}{\pi}\left[\int_{0}^{\pi} \frac{\sin(n+r)\theta}{\sin\theta}d\theta + \int_{0}^{\pi} \frac{\sin(n-r)\theta}{\sin\theta}d\theta\right]. \qquad (2.1.17)$$

If we suppose that

$$I_n = \frac{1}{\pi}\int_{0}^{\pi} \frac{\sin(n\theta)}{\sin\theta}d\theta = \begin{cases} 0 & if \quad n\ is\ even, \\ 1 & if \quad n\ is\ odd. \end{cases} \qquad (2.1.18)$$

then equations (2.1.16) and (2.1.17) become

$$B_{1,n+1} = nI_n\ , \qquad (2.1.19)$$

$$B_{r+1,n+1} = n(I_{n+r} + I_{n-r}). \qquad (2.1.20)$$

Suppose that $j = r + 1$, $k = n + 1$, then equations (2.1.19) and (2.1.20) become

$$B_{1,k} = \begin{cases} 0 & if \quad k-1\ is\ even, \\ k-1 & if \quad k-1\ is\ odd, \end{cases}$$

$$B_{j,k} = \begin{cases} 0 & if \quad j+k-2 \quad is\ even, \\ 2(k-1) & if \quad j+k-2 \quad is\ odd, \end{cases}$$

which can be written as

$$B_{j,k} = (k-1)\begin{cases} 0 & if \quad j+k \quad is\ even\ , \\ 1 & if \quad j+k\ is\ odd\ and\ j=1, \\ 2 & if \quad j+k\ \ is\ odd\ and\ j \neq 1, \end{cases}$$

and this can be put in the form

$$B_{j,k} = \begin{cases} 0 & if \quad j+k\ \ is\ even, \\ (k-1)(2-\delta_{1j}) & if \quad j+k\ \ is\ odd. \end{cases} \tag{2.1.21}$$

From (2.1.11) and (2.1.12), we have

$$f'(x) = \sum_{n\sec\quad=0}^{\infty} \sum_{m=0}^{\infty} B_{m+1,n+1}\, a_{n+1}\, T_m(x),$$

and

$$f^r(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B^r_{m+1,n+1}\, a_{n+1}\, T_m(x).$$

2.2 Example

Here we shall use the method of Chebyshev polynomials to determine the eigenvalue of the classical Benard problem under the effect of rotation when the fluid layer is heated from below. The relative equations are (see Chandrasekhar [13])

$$\sigma \mathcal{L} w = \mathcal{L}^2 w - a^2 \sqrt{R}\, \theta \; - \sqrt{T}\, D\, \xi$$

$$\sigma P_r \theta = \mathcal{L}\theta + \sqrt{R}\, w \qquad (2.2.1)$$

$$(\mathcal{L} - \sigma)\xi = -\sqrt{T}\, Dw$$

where $D = \frac{\partial}{\partial x_3}$, $\mathcal{L} = \mathrm{D}^2 - a^2$, a is the wave number, R is the Rayleigh number, T is the Taylor number, P_r is the viscous Prandtle number, σ is the growth rate and w, ξ , θ are the third components of velocity, vorticity and temperature. To obtain accurate results and exclude any numerical divergence it is convenient to reduce the order of higher order terms, so let us reduce the order of the term $\mathcal{L}^2 w$ in equation $(2.2.1)_1$. Suppose that $\varphi = \mathcal{L}w$, then equations (2.2.1) become

$$\sigma\varphi = \mathcal{L}\, \varphi - a^2 \sqrt{R}\, \theta - \sqrt{T}\, D\, \xi,$$

$$\sigma P_r \theta = \mathcal{L}\theta + \sqrt{R}\, w, \qquad (2.2.2)$$

$$\sigma\xi = \mathcal{L}\xi + \sqrt{T}\, Dw,$$

$$\sigma \times 0 = \mathcal{L}w - \varphi\, .$$

The system of equations (2.2.2) can be solved using free or rigid boundaries. For free boundaries the conditions are

$$\varphi = w = \theta = D\xi = 0\,, \qquad x_3 = 0, 1 \qquad (2.2.3)$$

and for rigid boundaries the conditions are

$$w = Dw = \theta = \xi = 0\,. \qquad x_3 = 0, 1 \qquad (2.2.4)$$

Since the Chebyshev polynomials are defined in the interval $[-1,1]$ and our problem is defined in the interval $[0,1]$, then we may introduce the variable x such that

$$x_3 = \frac{1}{2}[(1-0)x + (1+0)] = \frac{1}{2}(x+1)$$

$$\therefore \; x = 2x_3 - 1$$

Also,

$$D(\,) = \frac{d(\,)}{dx_3} = \frac{d(\,)}{dx}\frac{dx}{dx_3} = \frac{d(\,)}{dx}\frac{d(2x_3-1)}{dx_3} = 2\frac{d(\,)}{dx} = 2B(\,),$$

$$D^2(\,) = \frac{d^2(\,)}{d{x_3}^2} = \frac{d(\,)}{dx_3}\frac{d(\,)}{dx_3} = 2\frac{d(\,)}{dx}2\frac{d(\,)}{dx} = 4B^2(\,),$$

$$\mathcal{L}(\,) = (D^2 - a^2)(\,) = (4B^2 - a^2)(\,) = 4\left(B^2 - \left(\frac{a}{2}\right)^2\right)(\,) = 4V(\,),$$

where $V(\,) = B^2(\,) - \left(\frac{a}{2}\right)^2(\,)$ and $\underline{B}$ is the derivative matrix defined in (2.1.21). Now we express all the variables of the problem in terms of Chebyshev polynomials in the following way:

$$(\varphi, \theta, \xi, w) = \sum_{n=o}^{\infty}(c_{n+1}, b_{n+1}, d_{n+1}, a_{n+1})T_n\,(x). \qquad (2.2.5)$$

Apply (2.2.5) into the governing equations (2.2.2) and the boundary conditions (2.2.3) and (2.2.4), then the governing equations become

$$\sigma c_{n+1} = 4Vc_{n+1} - a^2\sqrt{R}\,b_{n+1} - 2\sqrt{T}Bd_{n+1},$$

$$\sigma P_r b_{n+1} = 4Vb_{n+1} + \sqrt{R}\,a_{n+1}, \qquad (2.2.6)$$

$$\sigma d_{n+1} = 4Vd_{n+1} + 2\sqrt{T}B\,a_{n+1},$$

$$\sigma \times 0 = 4Va_{n+1} - c_{n+1}.$$

The free boundary conditions become

$$\sum_{n=o}^{\infty} c_{n+1} T_n(x) = \sum_{n=0}^{\infty} b_{n+1} T_n(x) = \sum_{n=0}^{\infty} d_{n+1} T_n'(x) = 0,$$

$$\sum_{n=0}^{\infty} a_{n+1} T_n(x) \ = 0\,, \qquad\qquad x_3 = 0, 1. \qquad\qquad (2.2.7)$$

The rigid boundary conditions become

$$\sum_{n=o}^{\infty} a_{n+1} T_n(x) = \sum_{n=0}^{\infty} b_{n+1} T_n(x) = \sum_{n=0}^{\infty} d_{n+1} T_n(x) = 0,$$

$$\sum_{n=0}^{\infty} a_{n+1} T_n'(x) \ = 0, \qquad\qquad x_3 = 0, 1. \qquad\qquad (2.2.8)$$

Equations (2.2.6) can be written in the form

$$\sigma \underline{E}\ \underline{X} = \ \underline{F}\ \underline{X}$$

where

$$\underline{E} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & P_r I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \underline{F} = \begin{bmatrix} 4V & -a^2\sqrt{R}\, I & -2\sqrt{T}B & 0 \\ 0 & 4V & 0 & \sqrt{R}\, I \\ 0 & 0 & 4V & 2\sqrt{T}B \\ -I & 0 & 0 & 4V \end{bmatrix}$$

and $\underline{X} = [c_{n+1}\ b_{n+1}\ d_{n+1}\ a_{n+1}]^T$. Using (2.1.5) and (2.1.8), the free boundary conditions (2.2.7) become

$$\sum_{n=o}^{\infty} (\pm 1)^n c_{n+1} = \sum_{n=0}^{\infty} (\pm 1)^n b_{n+1} \ = \sum_{n=0}^{\infty} (\pm 1)^{n-1} n^2 d_{n+1} = 0,$$

$$\sum_{n=0}^{\infty} (\pm 1)^n a_{n+1} \ = 0. \qquad\qquad (2.2.9)$$

The rigid boundary conditions become

$$\sum_{n=o}^{\infty}(\pm 1)^n a_{n+1} = \sum_{n=0}^{\infty}(\pm 1)^n b_{n+1} = \sum_{n=0}^{\infty}(\pm 1)^n d_{n+1} = 0,$$

$$\sum_{n=0}^{\infty}(\pm 1)^{n-1} n^2 a_{n+1} \quad = 0. \qquad (2.2.10)$$

Notice that each element of the $\underline{E}$ and $\underline{F}$ matrices represent a square matrix. The order of this matrix depends on the number of Chebyshev polynomials used for required accuracy. In general the boundary conditions (2.2.9) or (2.2.10) should be inserted in the first and second rows of each diagonal element of the matrix $\underline{F}$. The corresponding rows in other elements of $\underline{E}$ and $\underline{F}$ must be set to zero.

Each diagonal element of $\underline{E}$ and $\underline{F}$ corresponds to a variable element in $\underline{X}$, so in this problem we can see from equations (2.2.6) that the first, the second the third and the fourth diagonal elements correspond to $\varphi, \theta, \xi \ and \ w$ respectively. Thus the first row of each diagonal element of $\underline{F}$ contains the condition of the corresponding variable at $x = 1,$ whereas the second row contains the condition of that variable at $x = -1$.

If the boundary conditions in (2.2.9) and (2.2.10) are of the form

$$\sum_{n=0}^{\infty}(\pm 1)^n \alpha_{n+1} = 0,$$

then the first row of the corresponding matrix is of the form $(1,1,1,\dots)$ and the second row is of the form $(1,-1,1,-1,\dots)$. However if the boundary conditions in (2.2.9) and (2.2.10) are of the form

$$\sum_{n=0}^{\infty}(\pm 1)^{n-1} \alpha_{n+1} n^2 = 0,$$

then the first row of the corresponding matrix is of the form $(0,1,4,9,16,\dots)$ and the second row is of the form $(0,-1,4,-9,16,\dots)$.

In the case of rigid boundaries, the condition $\varphi = 0$ is replaced by $Dw = 0$, so we remove the condition of $\varphi = 0$ which is written in the diagonal element of the first row of the matrix $\underline{F}$ and insert the condition $Dw = 0$ in the same row but in the column corresponds to the variable w, i.e. in the fourth column in this problem.

In practice, we cannot use infinite series of Chebyshev polynomials and we have to rely on a finite approximation of suitable accuracy, so we look for an approximation of the form

$$f(x) = \sum_{n=0}^{N} a_{n+1} T_n(x),$$

where N is the number of Chebyshev polynomials required.

In the theory of convection, we introduce time dependence through $\exp(\sigma t)$ and instability occurs if any eigenvalue σ has a positive real part. The most competitive or "least stable" eigenvalue has the algebraically largest real part. The eigenvalue problem (2.2.6) together with the boundary conditions (2.2.9) or (2.2.10) can be solved using a numerical routine F02BJF which is part of a mathematical library called Numerical Algorithm Group (NAG). The fortran codes for this example are listed in appendices 1 and 2.

2.3 The golden section search:

Here we shall explain a method used in the following chapter to obtain the minimum of the Rayleigh number as a function of the wave number. It deals with a unimodel function, i.e. a function which has only a single local minimum. This method is an iterative method and it has been termed the "golden section search" since it depends on a ratio r known to the early Greeks as the golden section ratio and is given by

$$r = \frac{\sqrt{5}-1}{2} \simeq 0.6180339887$$

where $r^2 = 1 - r$. (see Cheney and Kincaid [15])

Suppose that *G(x)* has a single minimum in the interval [*a* ,*b*]. In each step we need to replace this interval by a smaller one that is also contains the minimum point. Moreover in each step we need to specify two values of *G* at two particular points in [*a* ,*b*] i.e.

$$x = a + r^2(b-a) \qquad u = G(x),$$

$$y = a + r(b-a) \qquad v = G(y).$$

There are two cases to consider:

Case 1: $(u > v)$

Since *G* is unimodal, the minimum of *G* must be in the interval $[x, b]$ and so in the next step *G* is evaluated at x^*, y^* where

$$x^* = x + r^2(b-x),$$

$$y^* = x + r\ (b-x).$$

We notice that

$$x^* = a + r^2(b-a) + r^2[b - a - r^2(b-a)]$$

$$= a + r^2(b-a) + r^2(b-a) - r^4(b-a)$$

$$= a + (2r^2 - r^4)(b-a)$$

$$= a - (r^4 - 2r^2)(b-a)$$

$$= a - (r^4 - 2r^2 + 1 - 1)(b-a)$$

$$= a - [(r^2-1)^2 - 1](b-a)$$

Since $(r^2-1)^2 = (1-r^2)^2$ and $r^2 = 1 - r$, then

$$x^* = a + [1 - (1-r^2)^2](b-a)$$

$$= a + [1 - (1-(1-r))^2](b-a)$$

$$= a + [1 - r^2](b-a)$$

$$= a + r(b-a)$$

$$= y.$$

Thus

$$G(x^*) = G(y) = v.$$

Case 2: $(u \leq v)$

The minimum of G must be in the interval $[a, y]$ and so G is evaluated at x^*, y^* where

$$x^* = a + r^2(y-a),$$

$$y^* = a + r\,(y-a).$$

We notice that

$$y^* = a + r[a + r(b-a) - a]$$

$$= a + r[r(b-a)]$$

$$= a + r^2(b-a)$$

$$= x.$$

Thus

$$G(y^*) = G(x) = u.$$

In both cases the value of *G* at one of the new points is already known. Also we notice that

$$|y-a| = |b-x| = b - a - r^2(b-a)$$

$$= (b-a)(1-r^2)$$

$$= r(b-a),$$

so that after *N* iterations of the method, the interval [a ,b] is reduced to one of length $r^N(b-a)$ and so if ε is the required accuracy then we need to choose *N* such that

$$r^N(b-a) < \varepsilon$$

i.e.

$$r^N < \frac{\varepsilon}{(b-a)}$$

and so

$$\log r^N < \log \frac{\varepsilon}{(b-a)}$$

i.e.

$$N \simeq \frac{\log \frac{\varepsilon}{(b-a)}}{\log r}.$$

Chapter 3

CHAPTER THREE

Thermal convection in a horizontal porous layer affected by rotation

3.1 The basic equations

The basic equations governing the hydrodynamical flow of a viscous fluid of varying density and temperature are

i. Equation of continuity

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v_i)}{\partial x_i}=0, \tag{3.1.1}$$

where ρ is the density and v_i is the velocity. For an incompressible fluid $\rho=\rho_0$, where ρ_0 is a constant, and equation (3.1.1) is reduced to

$$\frac{\partial v_i}{\partial x_i}=0. \tag{3.1.2}$$

ii. Equation of motion

$$\rho\frac{\partial v_i}{\partial t}+v_j\rho\frac{\partial v_i}{\partial x_j}=\rho b_i+\hat{\sigma}_{ij,j}\,, \tag{3.1.3}$$

where b_i represents the external forces and $\hat{\sigma}_{ij}$ is the extra stress tensor. Notice that

$$\hat{\sigma}_{ij}=-p\delta_{ij}+2\,\mu\,d_{ij}-\frac{2}{3}\mu\,d_{kk}\delta_{ij}\,, \tag{3.1.4}$$

where p is the isotropic pressure, μ is the coefficient of viscosity and

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$$

is the rate of strain. Substituting for $\hat{\sigma}_{ij}$ in equation (3.1.3), we have

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = \rho b_i - p_{,i} + \frac{\partial}{\partial x_j}\left(\mu\,(v_{i,j} + v_{j,i}) - \frac{2}{3}\mu\, v_{k,k}\delta_{ij}\right). \quad (3.1.5)$$

For an incompressible fluid μ is constant and equation (3.1.5) is simplified to

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = \rho b_i - p_{,i} + \mu\nabla^2 v_i\,, \quad (3.1.6)$$

which are the Navier – Stokes equations.

iii. The heat conduction equation

$$\rho\frac{\partial(c_v T)}{\partial t} + \rho v_j\frac{\partial(c_v T)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) - p\frac{\partial v_j}{\partial x_j} + \Phi, \quad (3.1.7)$$

where c_v is the specific heat at constant volume, T is the temperature and k is the coefficient of heat conduction and

$$\Phi = 2\,\mu\,(d_{ij})^2 - \frac{2}{3}\mu\,(d_{jj})^2, \quad (3.1.8)$$

is the viscous dissipation. For an incompressible fluid equation (3.1.7) is reduced to

$$\rho\frac{\partial(c_v T)}{\partial t} + \rho v_j\frac{\partial(c_v T)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) + 2\,\mu\,(d_{ij})^2. \quad (3.1.9)$$

3.2 Formulation of the problem:

Consider an infinite horizontal layer occupied by a porous medium permeated by an incompressible viscous fluid. The fluid is subject to a constant gravitational acceleration $\underline{g}$ in the negative x_3 direction. The fluid layer is rotated about the x_3 axis at a constant angular velocity $\underline{\Omega}$ (see figure 1).

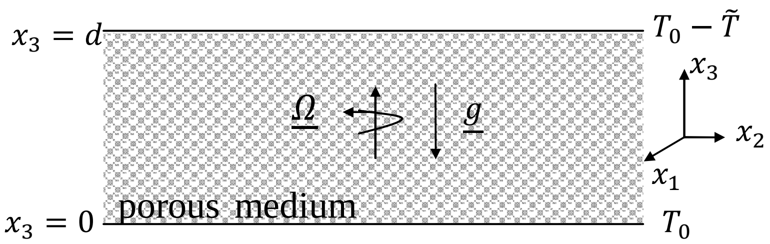

Figure 1.

The governing equations of the problem are (3.1.2), (3.1.6) and (3.1.9). These equations must be supplemented by an equation of state. This equations depends on the Boussinesq approximation which state that density is constant everywhere except in the body force term in the equation of motion where the density is linearly proportional to temperature, i.e.

$$\rho = \rho_0 \left[1 - \alpha(T - T_0)\right], \tag{3.2.1}$$

where α is the coefficient of volume expansion and T_0 is the temperature at $x_3 = 0$ and $T_0 - \tilde{T}$ is the temperature at $x_3 = d$, where $\tilde{T} = \beta d$. Since the fluid is rotated about the x_3 axis with a constant angular velocity $\underline{\Omega}$, and the porous medium is controlled by Brinkman model then the equation of motion (3.1.6) becomes

$$\dot{v}_i = -\left(\frac{p}{\rho_0}\right)_{,i} + \nu\nabla^2 v_i - g[1 - \alpha(T - T_0)]\delta_{i3} - \frac{\nu}{k_1} v_i$$

$$+2(\underline{v} \times \underline{\Omega})_i + \frac{1}{2}\left|\underline{\Omega} \times \underline{r}\right|^2_{,i}, \tag{3.2.2}$$

where

$$\dot{v}_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j},$$

$\nu = \frac{\mu}{\rho_0}$ is the kinematic viscosity, k_1 is the permeability of porous medium, $2(\underline{v} \times \underline{\Omega})_i$ is the Coriolis acceleration, $\frac{1}{2}\left|\underline{\Omega} \times \underline{r}\right|^2_{,i}$ is the centrifugal force, $\underline{r}$ is the position vector and g is the gravity acceleration.

We shall assume that α, c_v and k are constants, so equations (3.1.2), (3.2.2) and (3.1.9) become

$$v_{i,i} = 0, \tag{3.2.3}$$

$$\dot{v}_i = -\left(\frac{P}{\rho_0}\right)_{,i} + \nu\nabla^2 v_i - g\big(1 - \alpha(T - T_0)\big)\delta_{i3} - \frac{\nu}{k_1} v_i$$

$$+2\epsilon_{ijk} v_j \Omega_k \,, \tag{3.2.4}$$

$$\dot{T} = \kappa \frac{\partial^2 T}{\partial x_j^2}, \tag{3.2.5}$$

where $\kappa = \frac{k}{\rho c_v}$ is the coefficient of thermal conductivity, ϵ_{ijk} is the permutation tensor and

$$P = p - \frac{\rho_0}{2}\left|\underline{\Omega} \times \underline{r}\right|^2,$$

is the modified pressure. Notice that the last term in equation (3.1.9) is ignored since the term is very small comparing to the other terms in the equation.

3.3 The perturbation equations

Suppose that the horizontal layer of fluid has a steady adverse temperature gradient and that there is no motion, then the initial state is that

$$v_i = 0, \qquad T = T(x_3), \tag{3.3.1}$$

so from equation (3.2.4) we have

$$P = P(x_3)\,, \tag{3.3.2}$$

and equation (3.2.5) becomes

$$\frac{\partial^2 T}{\partial x_j^2} = 0$$

and so

$$\frac{\partial T}{\partial x_j} = -\beta$$

and

$$T = T_0 - \beta x_3, \tag{3.3.3}$$

where β is the adverse temperature gradient.

Now let the initial state described by (3.3.1)-(3.3.3) be slightly perturbed such that

$$v_i = 0 + \varepsilon\, \hat{v}_i\,, \qquad T = T_0 - \beta x_3 + \varepsilon\hat{\theta}, \qquad P = P + \varepsilon\hat{P}.$$

where $\hat{v}_i, \hat{\theta}\,, \hat{P}$ are respectively the linear perturbation of velocity, temperature and pressure. So the linearized form of equations (3.2.3)-(3.2.5) are

$$\hat{v}_{i,i} = 0. \tag{3.3.4}$$

$$\frac{\partial \hat{v}_i}{\partial t} = -\left(\frac{\hat{P}}{\rho_0}\right)_{,i} + \nu \nabla^2 \hat{v}_i + g\,\alpha\,\hat{\theta}\,\delta_{i3} - \frac{\nu}{k_1}\hat{v}_i + 2\,\epsilon_{ijk}\,\hat{v}_j\,\Omega_k. \tag{3.3.5}$$

$$\frac{\partial \hat{\theta}}{\partial t} - \beta \hat{v}_3 = \kappa \nabla^2 \hat{\theta}. \tag{3.3.6}$$

Now we use the non-dimensional variables $x_i{}^*$, $v_i{}^*$, θ^*, P^* and t^* such that

$$t = \frac{d^2}{\nu}\, t^*, \qquad \hat{v}_i = \frac{\kappa}{d}\, v_i{}^*, \qquad x_i = d\, x_i{}^*,$$

$$\hat{\theta} = \frac{\kappa}{d}\sqrt{\frac{\nu|\beta|}{g\alpha\kappa}}\,\theta^* \text{ and } \hat{P} = \frac{\rho_0 \kappa \nu}{d^2}\, P^*$$

where d is the depth of the fluid layer, then equations (3.3.4)-(3.3.6) become

$$v_{i,i} = 0. \tag{3.3.7}$$

$$\frac{\partial v_i}{\partial t} = -P_{,i} + \nabla^2 v_i + \sqrt{R}\theta\delta_{i3} - \frac{1}{N}\, v_i + \sqrt{T_a}\,\epsilon_{ijk} v_j \delta_{k3}. \tag{3.3.8}$$

$$P_r \frac{\partial \theta}{\partial t} + H\sqrt{R}\, v_3 = \nabla^2 \theta. \tag{3.3.9}$$

where the (*) superscript has been dropped but all the variables are non-dimensional. Notice that the non-dimensional numbers R, P_r , T and N are given by:

$$R = \frac{d^4 g\alpha|\beta|}{\nu\,\kappa}, \qquad P_r = \frac{\nu}{\kappa}, \qquad T_a = 4\frac{d^4\Omega^2}{\nu^2} \text{ and } N = \frac{k_1}{d^2}$$

where R is the Rayleigh number, P_r is the viscous Prandtle number and T_a is the Taylor number and where

$$H = \frac{-\beta}{|\beta|} = \begin{cases} 1 & \text{when heating from above .} \\ -1 & \text{when heating from below.} \end{cases}$$

3.4 The boundary conditions

The fluid is confined between the planes $x_3 = 0$ and $x_3 = d$ and on these planes, we need to specify mechanical and thermal conditions. Suitable mechanical conditions assume either a rigid boundary on which no slip occurs or a free boundary on which no tangential stresses act. Suitable thermal conditions assume either an insulating or a perfectly conducting boundary.

Mechanical conditions:

On either a free or a rigid boundary the normal component of velocity must be vanished on these boundaries i.e.

$$v_3 = 0 \qquad \text{on} \qquad x_3 = 0\,, d. \tag{3.4.1}$$

For a rigid boundary, the no slip condition implies that the horizontal components of the fluid velocity and all of x_1, x_2 partial derivatives of each component of the fluid velocity vanish. i.e.

$$v_1 = v_2 = 0 \quad \text{and} \quad v_{1,1} = v_{2,2} = 0, \tag{3.4.2}$$

and from the continuity equation (3.2.3) we obtain

$$v_{3,3} = 0. \tag{3.4.3}$$

If we introduce $\underline{\xi}$ to be the fluid vorticity then $\underline{\xi} = curl\ \underline{v}$ and the normal component of vorticity has the form

$$\xi_3 = v_{2,1} - v_{1,2}\,. \tag{3.4.4}$$

It follows from (3.4.2) that

$$\xi_3 = 0. \tag{3.4.5}$$

For a free boundary no tangential stresses act, and since

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\, d_{ij},$$

then

$$\sigma_{13} = \sigma_{23} = 0 \tag{3.4.6}$$

and so

$$d_{13} = d_{23} = 0$$

then $$v_{1,3} + v_{3,1} = v_{2,3} + v_{3,2} = 0. \tag{3.4.7}$$

and from (3.4.1) we obtain

$$v_{1,3} = v_{2,3} = 0. \tag{3.4.8}$$

Since the fluid is satisfying the continuity equation (3.2.3) then

$$v_{3,33} = 0. \tag{3.4.9}$$

Now taking the normal derivative of equation (3.4.4) and using (3.4.8) we obtain

$$\xi_{3,3} = 0. \tag{3.4.10}$$

Thermal conditions:

For a perfectly conducting boundary the temperatures of the boundary and impinging fluid match, whereas on a perfectly insulating boundary no heat transfer can take place between the fluid and the

surrounding hence the normal derivative of temperature is zero. In mathematical terms, the thermal conditions are:

$$\theta = \theta_{ext}\,, \qquad \text{on a conducting boundary,} \qquad (3.4.11)$$

$$\theta_{,3} = 0 \qquad \text{on an insulating boundary.} \qquad (3.4.12)$$

where θ_{ext} is the temperature of the region exterior to the fluid boundary.

3.5 The eigenvalue problem:

Now we shall discuss the linear stability of the steady solution (3.3.1)-(3.3.3). In order to do that we shall construct the related eigenvalue problem from equations (3.3.7)-(3.3.9) and the boundary conditions. In many convection problems the vector components parallel to the direction of gravity (i.e. the x_3 direction) play a central role and so we introduce the variables w and ξ such that $w = v_3$, $\xi = \xi_3$. When we take the curl of equation (3.3.8), we obtain

$$\frac{\partial \xi_i}{\partial t} = \nabla^2 \xi_i + \sqrt{R}\, \epsilon_{ijk} \theta_{,j} \delta_{k3} - \frac{1}{N}\, \xi_i + \sqrt{T_a}\, v_{i,3}. \qquad (3.5.1)$$

Taking the curl of equation (3.5.1) we obtain

$$\frac{\partial}{\partial t} \nabla^2 v_i = \nabla^4 v_i + \sqrt{R}(\nabla^2 \theta \delta_{i3} - \theta_{,i3}) - \frac{1}{N}\, \nabla^2 v_i - \sqrt{T_a}\, \xi_{i,3}. \qquad (3.5.2)$$

The third components of equations (3.5.1), (3.5.2) and (3.3.9) are:

$$\frac{\partial \xi}{\partial t} = \nabla^2 \xi - \frac{1}{N}\, \xi + \sqrt{T_a}\, w_{,3}, \qquad (3.5.3)$$

$$\frac{\partial}{\partial t} \nabla^2 w = \nabla^4 w + \sqrt{R} \left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} \right) \theta - \frac{1}{N}\, \nabla^2 w - \sqrt{T_a}\, \xi_{,3}, \qquad (3.5.4)$$

$$P_r \frac{\partial \theta}{\partial t} + H\sqrt{R}\, w = \nabla^2 \theta. \qquad (3.5.5)$$

Now we look for a solution of the form

$$\varphi = \varphi(x_3) e^{i(nx_1 + mx_2) + \sigma t},$$

where n, m are the wave numbers of the harmonic disturbance and σ is the growth rate. Thus equations (3.5.3)-(3.5.5) become

$$\left(\mathcal{L} - \sigma - \frac{1}{N}\right) \xi = -\sqrt{T_a}\, Dw, \qquad (3.5.6)$$

$$\left(\mathcal{L} - \sigma - \frac{1}{N}\right) \mathcal{L}w - a^2 \sqrt{R}\, \theta \; - \sqrt{T_a}\, D\, \xi = \; 0, \qquad (3.5.7)$$

$$(\mathcal{L} - \sigma P_r)\theta = H\sqrt{R}\, w, \qquad (3.5.8)$$

where $D = \frac{\partial}{\partial x_3}$, $\mathcal{L} = D^2 - a^2$ and $a = \sqrt{n^2 + m^2}$ is the wave number. We shall eliminate ξ and θ from equation (3.5.7), to obtain

$$(\mathcal{L} - \sigma P_r)\left(\mathcal{L} - \sigma - \frac{1}{N}\right)^2 \mathcal{L}w - a^2 RH \left(\mathcal{L} - \sigma - \frac{1}{N}\right) w$$

$$+ T_a(\mathcal{L} - \sigma P_r)\, D^2 w = 0, \qquad (3.5.9)$$

which can be written as:

$$\sigma^3 P_r\, \mathcal{L}w - \sigma^2 \left[(1 + 2P_r)\mathcal{L}^2 - \frac{2P_r}{N}\, \mathcal{L}\right] w + \sigma \left[(2 + P_r)\mathcal{L}^3 - \frac{2}{N}(1 + P_r)\mathcal{L}^2\right.$$

$$\left. + \frac{P_r}{N^2}\mathcal{L} - a^2 RH + P_r T_a D^2\right] w + \left[-\mathcal{L}^4 + \frac{2\mathcal{L}^3}{N} - \frac{\mathcal{L}^2}{N^2} + (a^2 RH - T_a D^2)\mathcal{L}\right.$$

$$\left. - \frac{a^2 RH}{N}\right] w = 0. \qquad (3.5.10)$$

which is an eighth order ordinary differential equation to be satisfied by w.

3.6 The free boundary problem:

Here we shall consider both boundaries to be free but later on we shall present results for the corresponding rigid boundary value problem. For the free boundary value problem,

$$w = D^2 w = 0, \qquad on \quad x_3 = 0, 1. \tag{3.6.1}$$

Thus equation (3.5.10) has eigenfunctions $w = A\sin(l\pi z)$, where A is a constant and l is an integer, then

$$Dw = Al\pi \, \cos(l\pi z), D^2 w = -Al^2\pi^2 \sin(l\pi z) \, , \mathcal{L}w = -\lambda w$$

where $\lambda = l^2\pi^2 + \mathrm{a}^2$, and σ satisfies the cubic equation

$$\sigma^3 P_r + \sigma^2\left[(1+2P_r)\lambda + \frac{2P_r}{N}\right] + \sigma\left[(2+P_r)\lambda^2 + \frac{2(1+P_r)\lambda}{N} + \frac{P_r}{N^2}\right.$$

$$\left. + \frac{(a^2RH + P_rT_a l^2\pi^2)}{\lambda}\right] + \lambda^3 + \frac{2\lambda^2}{N} + \frac{\lambda}{N^2} + \frac{a^2RH}{N\lambda} + a^2RH + T_a\pi^2 l^2 = 0,$$

(3.6.2)

Since the coefficients of this polynomial are real then its solutions are either all real or one real and two complex conjugate pair solutions. In the former case instability happens if any real solution is positive and in the latter case if either the real solution is positive (i.e. stationary instability) or the real part of the complex solution is positive (i.e. overstability). The solutions of (3.6.2) are functions of P_r, N, T_a and R and we have to examine how the nature of these solutions depends on these variables by considering the following cases.

Case(1): When the fluid is heated from above

Here $H = 1$. So equation (3.6.2) has the form

$$f(\sigma) = \sigma^3 P_r + \sigma^2\left[(1 + 2P_r)\lambda + \frac{2P_r}{N}\right] + \sigma\left[(2 + P_r)\lambda^2 + \frac{2(1 + P_r)\lambda}{N} + \right.$$

$$\left.\frac{P_r}{N^2} + \frac{(a^2R + P_rT_a l^2\pi^2)}{\lambda}\right] + \lambda^3 + \frac{2}{N}\lambda^2 + \frac{\lambda}{N^2} + \frac{a^2R}{N\lambda} + a^2R + T_a\pi^2 l^2 = 0$$

We need to discuss the roots of the polynomial equation $f(\sigma) = 0$. Clearly all the coefficients of $f(\sigma)$ are positive and real. Thus $f(\sigma) = 0$ has either three negative real solutions or one negative real solution and two complex conjugate solutions. To show how that the real part of the complex conjugate solutions is negative let $\sum$ be the sum of the roots of $f(\sigma) = 0$ then

$$\sum = -\frac{1}{P_r}\left[(1 + 2P_r)\lambda + \frac{2P_r}{N}\right]$$

and we can show that

$$f\left(\sum\right) = -\frac{1}{P_r^2}\left[(1 + 2P_r)\lambda + \frac{2P_r}{N}\right]^3 + \frac{1}{P_r^2}\left[(1 + 2P_r)\lambda + \frac{2P_r}{N}\right]^3 - \frac{1}{P_r}\left[\frac{P_r}{N^2}\right.$$

$$\left. + (2 + P_r)\lambda^2 + \frac{2}{N}(1 + P_r)\lambda + (a^2R + P_rT_a l^2\pi^2)\lambda^{-1}\right]\left[\frac{2P_r}{N}\right.$$

$$\left. + (1 + 2P_r)\lambda\right] + \lambda^3 + \frac{2}{N}\lambda^2 + \frac{\lambda}{N^2} + a^2R + T_a\pi^2 l^2 + \frac{a^2R}{N}\lambda^{-1},$$

$$= -\frac{(2+P_r)(1+2P_r)\lambda^3}{P_r} - \frac{2(2+P_r)\lambda^2}{N} - \frac{2(1+P_r)(1+2P_r)\lambda}{NP_r}$$

$$-\frac{4(1+P_r)\lambda}{N^2} - \frac{(a^2R + P_rT_a l^2\pi^2)(1+2P_r)}{P_r} - \frac{(1+2P_r)\lambda}{N^2} - \frac{2}{N^3}$$

$$-\frac{2(a^2R + P_rT_a l^2\pi^2)}{N\lambda} + \lambda^3 + \frac{2}{N}\lambda^2 + \frac{\lambda}{N^2} + a^2R + T_a\pi^2 l^2 + \frac{a^2R}{N\lambda}.$$

$$= -\lambda^3\left(\frac{2P_r^2 + 4P_r + 2}{P_r}\right) - \lambda^2\left(\frac{6P_r^2 + 8P_r + 2}{NP_r}\right) - \lambda\frac{(6P_r + 4)}{N^2}$$

$$-\frac{2}{N^3} - \left(\frac{2P_r^2 l^2\pi^2 T_a + a^2RP_r + a^2R}{P_r}\right) - \left(\frac{a^2R + 2P_r l^2\pi^2 T_a}{\lambda N}\right),$$

from which it is obvious that $f(\sum) < 0$. Since $f'(\sigma) > 0$ then $f(\sigma)$ has a negative real root, σ_{real} , which is greater than $\sum$. Let σ_1 , σ_2 be the two complex conjugate roots of $f(\sigma)$ such that

$$\sigma_1 = \alpha + i\beta\,, \qquad \sigma_2 = \alpha - i\beta,$$

then

$$\sum = \sigma_{real} + 2\alpha,$$

but $$\sum < \sigma_{real} < 0,$$

then $$\sum - \sigma_{real} < 0,$$

so $$2\alpha < 0 \quad \Rightarrow \quad \alpha < 0.$$

Thus if $f(\sigma)$ has only one real root and two complex conjugate roots then the real part of these conjugate solutions is negative. Hence when this fluid is heated from above, no instabilities ensue since all the roots are either negative if they are real or have negative real part if they are complex.

Case(2): When the fluid is heated from below

Here $H = -1$, so equation (3.6.2) has the form

$$\sigma^3 P_r + \sigma^2 \left[(1+2P_r)\lambda + \frac{2P_r}{N}\right] + \sigma\left[(2+P_r)\lambda^2 + \frac{2(1+P_r)\lambda}{N} + \frac{P_r}{N^2}\right.$$

$$\left. + \frac{(-a^2R + P_r T_a l^2 \pi^2)}{\lambda}\right] + \lambda^3 + \frac{2\lambda^2}{N} + \frac{\lambda}{N^2} - \frac{a^2 R}{N\lambda} - a^2 R + T_a \pi^2 l^2 = 0.$$

(3.6.3)

We shall discuss the stability of the fluid for the cases of stationary and overstability by producing the critical Rayleigh number for each case.

Stationary convection case:

To determine the critical Rayleigh number for the onset of stationary convection we set $\sigma = 0$ in equation (3.6.3). Thus

$$\lambda^3 + \frac{2}{N}\lambda^2 + \frac{\lambda}{N^2} - a^2 R + T_a \pi^2 l^2 - \frac{a^2 R}{N\lambda} = 0$$

$$\therefore \qquad R = \frac{\lambda}{a^2}\left(\lambda C + \frac{T_a \pi^2 l^2}{C}\right) \qquad (3.6.4)$$

where $C = \lambda + \frac{1}{N}$. The critical Rayleigh number can be obtained by minimizing R over the wave number a for several values of N and T_a.

To study the effect of permeability of porous medium and rotation on R, we find the derivative of each one from equation (3.6.4). Thus

$$\frac{dR}{dN} = -\frac{2\lambda^2}{a^2N^2} + \frac{\lambda}{a^2N^2C^2}(\lambda C^2 + T_a\pi^2 l^2)$$

$$= \frac{\lambda}{a^2N^2}\left(-\lambda + \frac{T_a\pi^2 l^2}{C^2}\right), \qquad (3.6.5)$$

$$\frac{dR}{dT_a} = \frac{\pi^2 l^2 \lambda}{a^2 C}, \qquad (3.6.6)$$

which means that rotation has a stabilizing effect on the system, and the permeability of porous medium has a stabilizing effect on the system if $T_a > \frac{\lambda C^2}{\pi^2 l^2}$.

The case of overstability:

To obtain the critical Rayleigh number for the overstability case we suppose that

$$x = \frac{a^2}{\pi^2}, i\sigma_1 = \frac{\sigma}{\pi^2}, R_1 = \frac{R}{\pi^4}, N_1 = N\pi^2, l = 1 \text{ and } T_{a1} = \frac{T_a}{\pi^4}. \qquad (3.6.7)$$

where σ is complex and $\sigma_1 \neq 0$. Thus equation (3.6.3) becomes:

$$-i\sigma_1^3 P_r - \sigma_1^2\left[(1+2P_r)(1+x) + \frac{2P_r}{N_1}\right] + i\sigma_1\left[(2+P_r)(1+x)^2 + \frac{P_r}{N_1^2}\right.$$

$$\left. + \frac{2(1+P_r)(1+x)}{N_1} + \frac{(P_r T_{a1} - xR_1)}{(1+x)}\right] + (1+x)^3 + \frac{2}{N_1}(1+x)^2$$

$$+ \frac{(1+x)}{N_1^2} - \frac{xR_1}{N_1(1+x)} - xR_1 + T_{a1} = 0.$$

After some algebra this equation reduces to

$$R_1 = (1 + x + P_r i\sigma_1)\left[\frac{1+x}{x}\left(1 + x + \frac{1}{N_1} + i\sigma_1\right)\right.$$

$$\left. + \frac{T_{a1}}{x}\,\frac{(1 + x + 1/N_1 - i\sigma_1)}{(1 + x + 1/N_1)^2 + \sigma_1^2}\right]. \qquad (3.6.8)$$

Equating the real and imaginary parts of this equation we obtain the following pair of equations

$$R_1 = \frac{1}{x}\left[(1+x)^3 - (1+x)\,P_r\sigma_1^2 + \frac{(1+x)^2}{N_1}\right.$$

$$\left. + \,T_{a1}\;\frac{(1+x)^2 + (1+x)/N_1 + P_r\sigma_1^2}{(1 + x + 1/N_1)^2 + \sigma_1^2}\right] \qquad (3.6.9)$$

and

$$(1+x)^2(1 + P_r) + \frac{(1+x)\,P_r}{N_1} = T_{a1}\,\frac{(1+x)(1 - P_r) - P_r/N_1}{(1 + x + 1/N_1)^2 + \sigma_1^2}.$$

$$(3.6.10)$$

From equation (3.6.10) we obtain

$$\sigma_1^2 = T_{a1}\frac{(1+x)(1 - P_r) - P_r/N_1}{(1+x)^2(1 + P_r) + P_r(1+x)/N_1} - (1 + \mathrm{x} + 1/N_1)^2 \quad (3.6.11)$$

from which we conclude that in order for σ^2 to be positive we must have

$$T_{a1}\left[(1+x)(1 - P_r) - \frac{P_r}{N_1}\right] > \left(1 + \mathrm{x} + \frac{1}{N_1}\right)^2\left[(1+x)^2(1 + P_r)\right.$$

$$\left. + \frac{P_r(1+x)}{N_1}\right]. \qquad (3.6.12)$$

So in order to have overstability case the condition (3.6.12) must be satisfied. This condition can be satisfied provided

$(i)\ P_r < 1,$

$(ii)\ (1+x)(1-\ P_r) > \frac{P_r}{N_1}.$

Using the expression of σ_1^2 from (3.6.11) into (3.6.9) we obtain,

$$R_1 = \frac{1}{x}\Bigg[(1+x)^3 + \frac{(1+x)^2}{N_1} + T_{a1}$$

$$-\left(\frac{(1+x)^2(1+2P_r)/N_1^2 + (1+x)^3(1+P_r)/N_1 + P_r(1+x)/N_1^3}{(1+x)(1-P_r) -\ P_r/N_1}\right)$$

$$+\left(T_{a1}\ \frac{(1+x)(1-P_r) -\ P_r/N_1}{(1+x)^2(1+P_r) + P_r(1+x)/N_1} - (1+x+1/N_1)^2\right)$$

$$\left(\frac{(1+x)^2(1-P_r^2) + (1+x)(P_r - P_r^2)/N_1}{(1+x)(1-P_r) -\ P_r/N_1} - (1+x)P_r\right)\Bigg]. \quad (3.6.13)$$

To study the effect of permeability of porous medium and rotation on R_1 for the overstability case we differentiate equation (3.6.8) with respect to N_1 and T_{a1} respectively. Thus

$$\frac{dR_1}{dN_1} = \frac{-1}{N_1^2}\Bigg\{(1+x)\left[\frac{(1+x)}{x} + \frac{T_{a1}}{x((1+x+1/N_1)^2+\sigma_1^2)}\right.$$

$$\left. -\frac{2T_{a1}(1+x+1/N_1)^2}{x((1+x+1/N_1)^2+\sigma_1^2)^2}\right] - \frac{2T_{a1}P_r\sigma_1^2(1+x+1/N_1)}{x((1+x+1/N_1)^2+\sigma_1^2)^2}$$

$$+i\sigma_1\left[P_r\left[\frac{1+x}{x} + \frac{T_{a1}}{x((1+x+1/N_1)^2+\sigma_1^2)}\right.\right.$$

$$\left.\left. -\frac{2T_{a1}(1+x+1/N_1)^2}{x((1+x+1/N_1)^2+\sigma_1^2)^2}\right] + \frac{2T_{a1}(1+x)(1+x+1/N_1)}{x((1+x+1/N_1)^2+\sigma_1^2)^2}\right]\Bigg\}. \quad (3.6.14)$$

Equating separately the imaginary and real parts we obtain

$$P_r\left[\frac{1+x}{x}+\frac{T_{a1}}{x((1+x+1/N_1)^2+\sigma_1^2)}-\frac{2T_{a1}(1+x+1/N_1)^2}{x((1+x+1/N_1)^2+\sigma_1^2)^2}\right]$$

$$=-\frac{2T_{a1}(1+x)(1+x+1/N_1)}{x((1+x+1/N_1)^2+\sigma_1^2)^2} \tag{3.6.15}$$

and

$$\frac{dR_1}{dN_1}=\frac{-1}{{N_1}^2}\left\{(1+x)\left[\frac{(1+x)}{x}+\frac{T_{a1}}{x((1+x+1/N_1)^2+\sigma_1^2)}\right.\right.$$

$$\left.\left.-\frac{2T_{a1}(1+x+1/N_1)^2}{x((1+x+1/N_1)^2+\sigma_1^2)^2}\right]-\frac{2T_{a1}P_r\sigma_1^2(1+x+1/N_1)}{x((1+x+1/N_1)^2+\sigma_1^2)^2}\right\} \tag{3.6.16}$$

from which we obtain

$$\frac{dR_1}{dN_1}=\frac{2T_{a1}}{N_1^2}\left[\frac{(1+x)^2(1+x+1/N_1)+(1+x+1/N_1)P_r^2\sigma_1^2}{xP_r((1+x+1/N_1)^2+\sigma_1^2)^2}\right]>0\,, \tag{3.6.17}$$

i.e. the permeability of porous medium has a stabilizing effect on the system in the overstability case.

Now

$$\frac{dR_1}{dT_{a1}}=\left[\frac{(1+x)(1+x+1/N_1)+P_r\sigma_1^2}{x((1+x+1/N_1)^2+\sigma_1^2)}\right]$$

$$+i\sigma_1\left[\frac{P_r(1+x+1/N_1)-(1+x)}{x((1+x+1/N_1)^2+\sigma_1^2)}\right]. \tag{3.6.18}$$

Equating separately the imaginary and real parts we obtain

$$P_r(1+x+1/N_1)=(1+x), \tag{3.6.19}$$

and

$$\frac{dR_1}{dT_{a1}} = \frac{(1+x)(1+x+1/N_1)+P_r\sigma_1^2}{x((1+x+1/N_1)^2+\sigma_1^2)}, \tag{3.6.20}$$

from which we obtain

$$\frac{dR_1}{dT_{a1}} = \frac{(1+x)^2+P_r^2\sigma_1^2}{P_r x((1+x+1/N_1)^2+\sigma_1^2)} > 0, \tag{3.6.21}$$

i.e. the rotation has a stabilizing effect on the system in the overstability case.

3.7 Numerical solutions of the eighenvalue problem:

The eigenvalue problem (3.5.6)-(3.5.8) together with the boundary conditions are to be solved numerically for the case when the fluid layer is heated from below using expansion of Chebyshev polynomials. Equations (3.5.6)-(3.5.8) can be written in the form

$$\sigma\mathcal{L}\, w = \mathcal{L}^2 w - \frac{1}{N}\,\mathcal{L}\, w - a^2\sqrt{R}\,\theta - \sqrt{T_a}\,D\,\xi\,,$$

$$\sigma\, P_r\theta = \mathcal{L}\theta + \sqrt{R}\,w, \tag{3.7.1}$$

$$\sigma\,\xi = \mathcal{L}\xi - \frac{1}{N}\xi + \sqrt{T_a}\,Dw.$$

To obtain accurate results and exclude any numerical divergence it is convenient to reduce the order of higher order terms in equations (3.7.1) so to reduce the order of the term $\mathcal{L}^2 w$ in equation $(3.7.1)_1$ we suppose that

$$\varphi = \mathcal{L}w.$$

Substitute for φ in equations (3.7.1) we obtain

$$\sigma\,\varphi = \left(\mathcal{L} - \frac{1}{N}\right)\varphi - a^2\sqrt{R}\,\theta - \sqrt{T_a}\,D\xi,$$

$$\sigma\, P_r\theta = \mathcal{L}\theta + \sqrt{R}\,w, \tag{3.7.2}$$

$$\sigma\,\xi = \left(\mathcal{L} - \frac{1}{N}\right)\xi + \sqrt{T_a}\,Dw,$$

$$\sigma \times 0 = \mathcal{L}w - \varphi.$$

From equations (3.4.1) and (3.4.9)-(3.4.11) the free boundary conditions become

$$\varphi = w = \theta = D\xi = 0\,, \qquad x_3 = 0, 1 \tag{3.7.3}$$

and from equations (3.4.1), (3.4.3), (3.4.5) and (3.4.11) the rigid boundary conditions become

$$w = Dw = \theta = \xi = 0 . \qquad x_3 = 0,1 \qquad (3.7.4)$$

Now we express all the variables of the problem in terms of Chebyshev polynomials in the following way

$$(\varphi, \theta, \xi, w) = \sum_{n=o}^{\infty} (c_{n+1}, b_{n+1}, d_{n+1}, a_{n+1}) T_n\,(x). \qquad (3.7.5)$$

Substitute into the governing equations (3.7.2) we obtain

$$\begin{aligned} \sigma c_{n+1} &= (4V-\chi)c_{n+1} - a^2\sqrt{R}\,b_{n+1} - 2\sqrt{T_a}\,Bd_{n+1}, \\ \sigma P_r b_{n+1} &= 4V b_{n+1} + \sqrt{R}\,a_{n+1}, \qquad (3.7.6) \\ \sigma d_{n+1} &= (4V-\chi)d_{n+1} - 2\sqrt{T_a}\,Ba_{n+1}, \\ \sigma \times 0 &= 4Va_{n+1} - c_{n+1}. \end{aligned}$$

where $\chi = N^{-1}$. The free boundary conditions (3.7.3) become

$$\sum_{n=o}^{\infty} c_{n+1} T_n(x) = \sum_{n=0}^{\infty} a_{n+1} T_n(x) = \sum_{n=0}^{\infty} b_{n+1} T_n(x) = \sum_{n=0}^{\infty} d_{n+1} T_n'(x) = 0$$

where $x = 0,1$ and the rigid boundary conditions (3.7.4) become

$$\sum_{n=o}^{\infty} a_{n+1} T_n(x) = \sum_{n=0}^{\infty} a_{n+1} T_n'(x) = \sum_{n=0}^{\infty} b_{n+1} T_n(x) = \sum_{n=0}^{\infty} d_{n+1} T_n(x) = 0$$

where $x = 0,1$. Equations (3.7.6) can be written in the form

$$\sigma \underline{E}\,\underline{X} = \underline{F}\,\underline{X}$$

where

$$\underline{E} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & P_r I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \underline{F} = \begin{bmatrix} 4V - \chi I & -a^2\sqrt{R}\, I & -2\sqrt{T_a}\, B & 0 \\ 0 & 4V & 0 & \sqrt{R}\, I \\ 0 & 0 & 4V - \chi I & 2\sqrt{T_a}\, B \\ -I & 0 & 0 & 4V \end{bmatrix},$$

$$(3.7.7)$$

and $\underline{X} = [c_{n+1}\ b_{n+1}\ d_{n+1}\ a_{n+1}]^T$. Using (2.1.5) and (2.1.8), the free boundary conditions become

$$\sum_{n=o}^{\infty} c_{n+1}(\pm 1)^n = \sum_{n=0}^{\infty} b_{n+1}\,(\pm 1)^n = \sum_{n=0}^{\infty} a_{n+1}(\pm 1)^n = 0,$$

$$\sum_{n=0}^{\infty} d_{n+1} n^2 (\pm 1)^{n-1} = 0, \qquad (3.7.8)$$

and the rigid boundary conditions become

$$\sum_{n=o}^{\infty} a_{n+1}(\pm 1)^n = \sum_{n=0}^{\infty} b_{n+1}(\pm 1)^n = \sum_{n=0}^{\infty} d_{n+1}(\pm 1)^n = 0,$$

$$\sum_{n=0}^{\infty} a_{n+1} n^2 (\pm 1)^{n-1} = 0. \qquad (3.7.9)$$

The eighenvalue problem (3.7.7) together with the boundary conditions (3.7.8) and (3.7.9) are solved using the numerical routine F02BJF of the NAG package. The fortran codes are listed in appendices 3 and 4.

3.8 Numerical Results and Discussion

The relation between the Taylor number, T_a, and the critical Rayleigh number, R, for different values of the non-dimensional permeability of porous medium, N, when both boundaries are free for the stationary convection case is displayed in figure (2). It is clear from the curves in the figure that as T_a increases R increases which indicates that rotation has a stabilizing effect on the system. Moreover as N decreases R increases provided that T_a is less than a certain value and when T_a exceeds that value then R decreases as N decreases which indicates that rotation changes the effect of the non-dimensional permeability of porous medium. In case of no porosity the critical Rayleigh number, R, is less than that of the porous medium case provided T_a is less than a certain value and when T_a exceeds that value then the critical Rayleigh number in the absence of porous medium case is higher.

Similarly the relation between the Taylor number, T_a, and the critical wave number, a, is displayed in figure (3) for different values of N when both boundaries are free for the stationary convection case. Clearly as T_a increases, a increases and as N decreases, a increases provided T_a is less than a certain value and when T_a exceeds that value then a decreases as N decreases. The numerical results related to figures (2) and (3) are listed in table (1).

In case of overstability, the relation between the Taylor number, T_a, and the critical Rayleigh number, R, when both boundaries are free for different values of N and different values of P_r is displayed in figures (4) and (5). It is clear from the curves in the figures that as T_a increases, R increases which indicates that rotation has a stabilizing effect on the system in this case also. Moreover as N decreases, R increases for all values of T_a.

i.e. the system becomes more stable as the porous medium permeability decreases. In case of no porosity the critical Rayleigh number, R, is less than the corresponding Rayleigh number in porous medium for all values of T_a.

Similarly the relation between the Taylor number, T_a, and the critical wave number, a, is displayed in figures (6) and (7) for different values of N when both boundaries are free for the overstability case. Clearly as T_a increases, a increases and as P_r increases, a increases for all values of T_a. The numerical results related to figures (4)-(7) are listed in tables (2) and (3).

It is important to remark that in absence of porous medium, which is the classical Benard problem under the effect of rotation, overstability appears when the Taylor number, T_a, exceeds a certain value and the Prandtl number $P_r < 1$. In this work this result is proved numerically in the presence of porous medium and we can see that clearly from figures (4) and (5) and from the numerical results listed in tables (2) and (3).

A comparison between the stationary convection and overstability cases when both boundaries are free is displayed in figures (8-10) for $N=0.1$, $N=0.01$ and $N=0.001$ respectively. It is clear from these figures that overstability is the preferred mechanism provided $P_r < 1$ and the Taylor number, T_a, exceeds a certain critical value. This critical value of T_a increases as N decreases. Moreover for the overstability case we notice that as P_r increases the critical Rayleigh number, R, increases which indicates that the Prandtl number has a stabilizing effect on the system.

Similarly the relation between T_a and the wave number, a, when both boundaries are free is displayed in figures (11-13) for $N=0.1$, $N=0.01$ and $N=0.001$ respectively. These figures show a comparison between

stationary convection and overstability cases. For the overstability case we notice that as P_r increases the wave number increases.

Numerical results are also obtained when both boundaries are rigid. In this case the relation between the Taylor number, T_a, and the critical Rayleigh number, R, for different values of the non-dimensional permeability of porous medium, N, for the stationary convection case is displayed in figure (14). It is clear from the curves in the figure that as T_a increases R increases which indicates that rotation has a stabilizing effect on the system. Moreover as N decreases R increases for all values of $T_a$and we notice here that this conclusion is different from that in the case when both boundaries are free. In case of no porosity the critical Rayleigh number, R, is always less than the corresponding Rayleigh number in porous medium for all values of T_a.

Similarly the relation between the Taylor number, T_a, and the critical wave number, a, is displayed in figure (15) for different values of N when both boundaries are rigid for the stationary convection case. Clearly as T_a increases, a increases and as N decreases, a decreases for all values of T_a. The numerical results related to figures (14) and (15) are listed in table (4).

In case of overstability, the relation between the Taylor number, T_a, and the critical Rayleigh number, R, when both boundaries are rigid for different values of N and different values of P_r is displayed in figures (16) and (17). It is clear from the curves in the figures that as T_a increases, R increases which indicates that rotation has a stabilizing effect on the system in this case also. Moreover as N decreases, R increases for all values of T_a. i.e. the system becomes more stable as the porous medium permeability decreases. In case of no porosity the critical Rayleigh number, R, is always

less than the corresponding Rayleigh number in porous medium for all values of T_a.

Similarly the relation between the Taylor number, T_a, and the critical wave number, a, is displayed in figures (18) and (19) for different values of N when both boundaries are rigid for the overstability case. Clearly as T_a increases, a increases and as N decreases, a increases for a wide range of $T_a T$. The numerical results related to figures (16)-(19) are listed in tables (5) and (6).

A comparison between the stationary convection and overstability cases when both boundaries are rigid is displayed in figures (20,21) for $N=0.01$ and $N=0.001$ respectively. It is clear from these figures that overstability is the preferred mechanism provided $P_r < 1$ and the Taylor number, T_a, exceeds a certain critical value. This critical value of T_a increases as N decreases. Moreover for the overstability case we notice that as P_r increases the critical Rayleigh number, R, increases which indicates that the Prandtl number has a stabilizing effect on the system.

Similarly the relation between T_a and the critical wave number, a, when both boundaries are rigid is displayed in figures (22,23) for $N=0.01$ and $N=0.001$ respectively. These figures show a comparison between stationary convection and overstability cases. For the overstability case we notice that as P_r increases the wave number increases.

A comparison between the numerical results of the free and rigid boundaries in the stationary convection case is shown in figures (24,25) for $N=0.01$ and $N=0.001$ respectively. Clearly the critical Rayleigh numbers, R, for the rigid boundary case are always greater than those of the free boundary case.

Similarly a comparison of the wave number, *a*, between the free and rigid boundaries in the stationary convection case is shown in figures (26,27) for *N=0.01* and *N=0.001* respectively. The wave numbers in the rigid boundary case are always greater than those of the free boundary case.

For the overstability case, a comparison between the numerical results for the free and rigid boundaries is displayed (when $P_r = 0.05$) in figures (28,29) for *N=0.01* and *N=0.001* respectively, and in figures (30,31) for *N=0.01* and *N=0.001* respectively (when $P_r = 0.025$). In all of these figures, the critical Rayleigh numbers, *R*, for the rigid boundary case are always greater than those of the free boundary case.

Similarly a comparison of the wave number, *a*, between the free and rigid boundaries in the overstability case is shown in figures (32,33) for *N=0.01* and *N=0.001* respectively (when $P_r = 0.05$) and in figures (34,35) for *N=0.01* and *N=0.001* respectively (when $P_r = 0.025$). In all of these figures, the wave numbers in the rigid boundary case are always greater than those of the free boundary case.

Table 1. The relation between T_a and R for the stationary stability case when both boundaries are free for different values of N.

T_a	NO POROUSITY		$N=0.1$		$N=0.01$		$N=0.001$	
	a	R	a	R	a	R	a	R
0	2.221	657.511	2.472	1085.734	2.923	4699.135	3.112	40253.990
10	2.270	677.077	2.509	1095.738	2.923	4700.932	3.112	40254.185
100	2.594	826.290	2.607	1180.548	2.929	4717.079	3.112	40255.944
300	3.011	1075.488	2.823	1349.728	2.940	4752.833	3.112	40259.853
1000	3.710	1676.118	3.318	1828.851	2.981	4876.630	3.113	40273.534
3000	4.554	2810.435	4.059	2833.789	3.089	5219.949	3.113	40312.609
10000	5.698	5377.142	5.183	5235.167	3.399	6330.580	3.119	40451.524
30000	6.961	10205.230	6.479	9873.539	4.007	9068.135	3.135	40838.193
100000	8.626	21309.002	8.202	20696.065	5.201	16737.804	3.186	42185.110
1000000	12.863	92223.612	12.556	90677.653	9.696	76893.577	3.693	58093.558
3000000	15.515	188370.825	15.256	186066.643	12.813	165102.267	4.394	88854.068
10000000	19.024	414698.263	18.810	411177.695	16.808	379051.138	5.703	181014.785
30000000	22.894	855438.722	22.715	850293.626	21.054	803435.119	7.618	409260.510
60500000	25.758	1360112.689	25.598	1353573.009	24.393	1197310.287	9.519	724313.039
100000000	28.024	1897035.482	27.877	1889276.364	26.525	1818818.681	11.546	1103062.709
200000000	31.476	3003563.809	31.344	2993749.970	30.143	2904771.685	15.871	1968935.230
300000000	33.687	3930829.970	33.564	3919574.951	32.443	3817612.853	19.150	2741130.116
605000000	37.882	6263059.270	37.772	6248801.599	36.776	6119797.164	25.319	4763684.720
700000000	38.817	6900383.575	38.710	6885408.460	37.771	6749961.119	26.636	5328135.324
805000000	40.073	7574148.173	39.630	7556275.554	38.639	7414350.144	27.907	5926704.044
1000000000	41.200	8746259.989	41.101	8729376.152	40.189	8576726.505	29.898	6981048.636

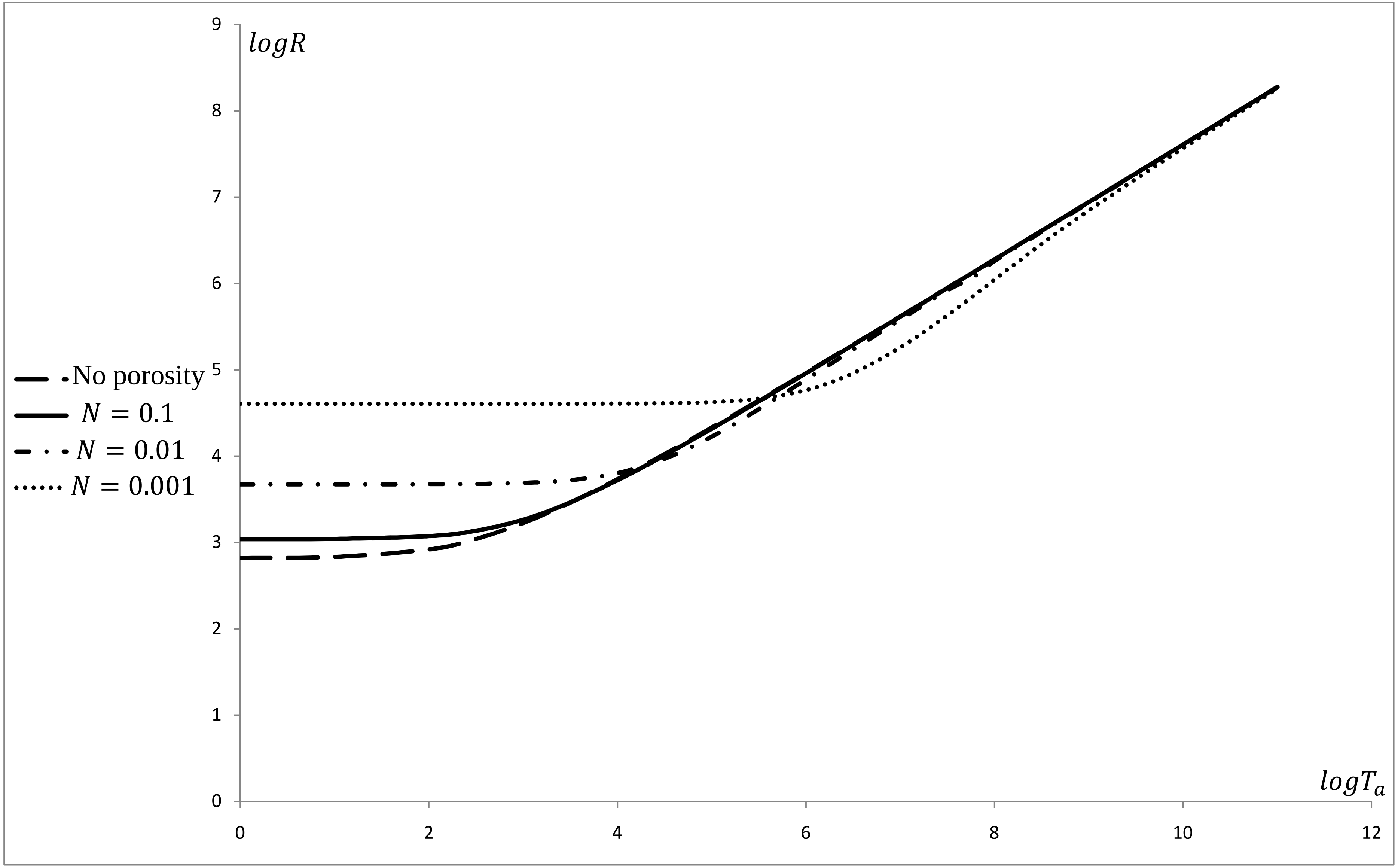

Figure 2. The relation between T_a and R for the stationary stability case when both boundaries are free for different values of N.

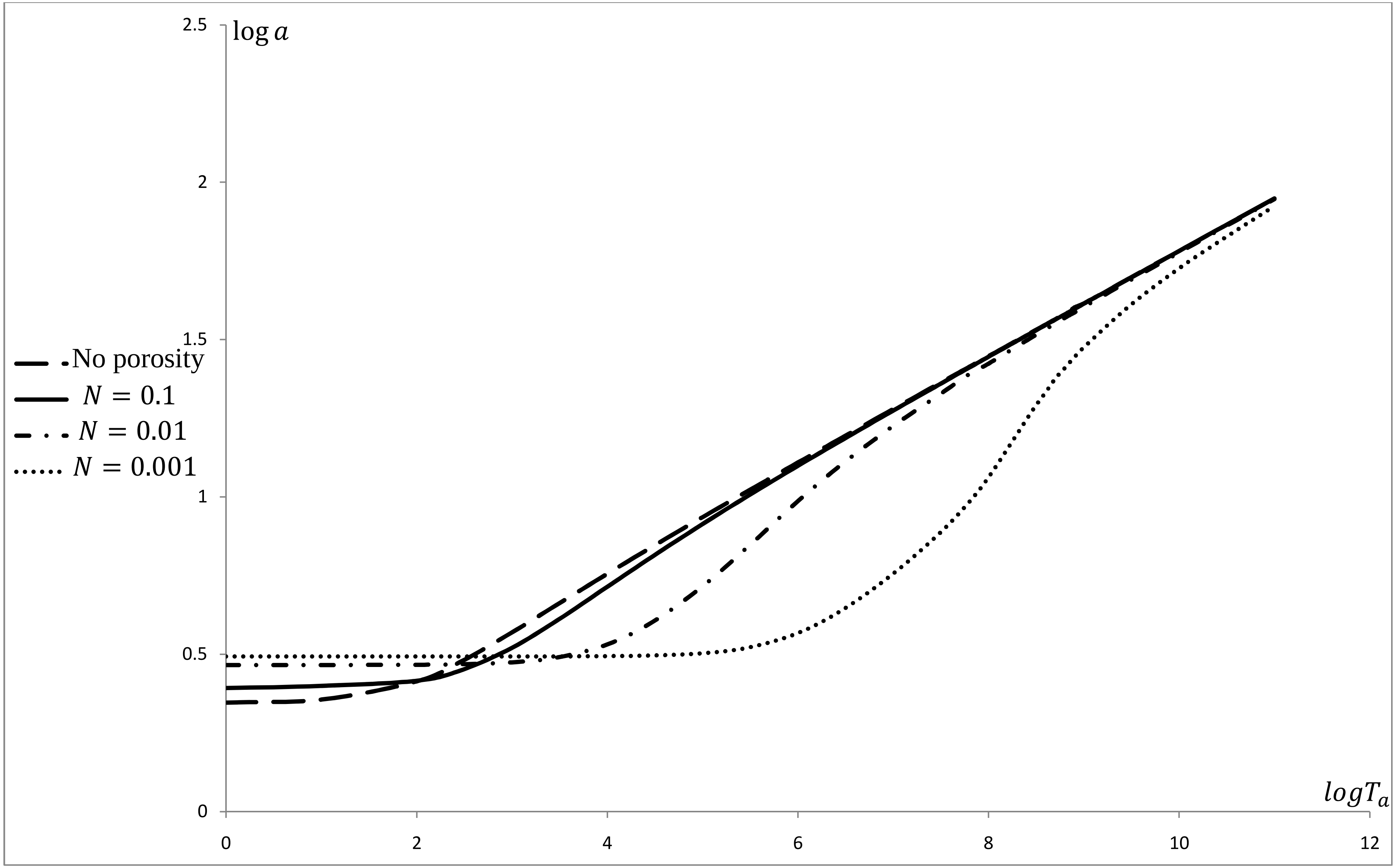

Figure 3. The relation between T_a and a for the stationary stability case when both boundaries are free for different values of N.

Table 2. The relation between T_a and R for the overstability case when both boundaries are free for different values of N. Here $P_r = 0.05$.

T_a	*NO POROUSITY*		*N=0.1*		*N=0.01*		*N=0.001*	
	a	R	a	R	a	R	a	R
0								
1000	2.233	1390.249						
3000	2.255	1408.917	2.875	2682.661				
10000	2.327	1471.581	2.879	2499.709				
30000	2.496	1632.935	2.921	2677.005				
100000	2.884	2083.909	3.143	3225.529	3.681	14143.329		
300000	3.460	3005.406	3.575	4367.729	4.177	16635.842		
1000000	4.332	5132.067	4.572	6872.557	5.102	22038.766		
10000000	6.618	18142.077	6.889	21870.808	7.496	48647.164		
30000000	8.058	35250.245	8.266	39809.226	8.983	77742.924		
60500000	9.119	54523.200	9.311	60204.993	10.063	107547.135		
100000000	9.956	74844.577	10.141	81511.616	10.972	137395.048	11.351	702625.020
200000000	11.073	118793.303	11.399	124735.251	12.360	194644.440	12.752	882584.926
300000000	11.927	154032.317	12.315	160939.773	13.234	240888.195	13.812	1015763.893
405000000	12.527	183562.819	13.002	194388.056	13.809	282412.506	14.538	1130754.655
500000000	13.082	214054.925	13.564	222350.032	14.356	316168.188	15.124	1221186.987
605000000	13.536	239778.507	14.130	249691.546	14.751	351764.189	15.844	1311136.952
700000000	13.938	261791.302	14.414	274331.496	15.099	380200.712	16.179	1387692.014
705000000	14.049	268726.267	14.518	276605.029	15.117	381577.800	16.328	1393758.540
805000000	14.483	297415.664	14.721	299848.486	15.537	410647.832	16.555	1469518.885
1000000000	15.127	344398.170	15.292	344413.649	16.042	464354.695	17.167	1587907.004

Table 3. The relation between T_a and R for the overstability case when both boundaries are free for different values of N. Here $P_r = 0.025$.

T_a	NO POROUSITY		*N=0.1*		*N=0.01*		*N=0.001*	
	a	*R*	*a*	*R*	*a*	*R*	*a*	*R*
0								
1000	2.224	1350.334						
3000	2.230	1355.186	2.492	2268.996				
10000	2.251	1371.967	2.512	2290.473				
30000	2.306	1418.332	2.566	2349.766				
100000	2.467	1566.005	2.708	2538.277	3.279	11522.149		
1000000	3.379	2786.119	3.504	4069.365	4.159	15110.680		
10000000	5.175	8262.786	5.430	10435.391	6.057	28493.830		
30000000	6.336	15383.178	6.622	18358.804	7.276	42856.229	7.461	308116.379
60500000	7.193	23339.494	7.422	27010.544	8.148	57338.354	8.474	370940.524
100000000	7.868	31687.346	8.088	35973.526	8.856	71280.547	9.279	428706.349
200000000	9.069	48750.718	9.101	54012.166	9.854	98036.758	10.500	530776.015
300000000	9.555	62799.621	9.747	68860.477	10.559	119061.413	11.274	605477.596
405000000	10.062	75924.325	10.255	82596.305	11.074	137981.731	11.878	669451.023
500000000	10.432	86800.806	10.624	93940.231	11.449	153308.293	12.317	719417.349
605000000	10.788	98015.399	10.964	105755.169	11.799	168841.560	12.726	768600.447
700000000	11.072	107591.385	11.243	115549.414	12.073	181921.570	13.046	809008.601
705000000	11.072	108082.298	11.257	116058.830	12.087	182588.263	13.062	811045.375
805000000	11.334	117665.469	11.511	125992.014	12.342	195525.444	13.359	850158.594
1000000000	11.765	135246.824	12.029	144213.243	12.771	218936.764	13.856	919115.075

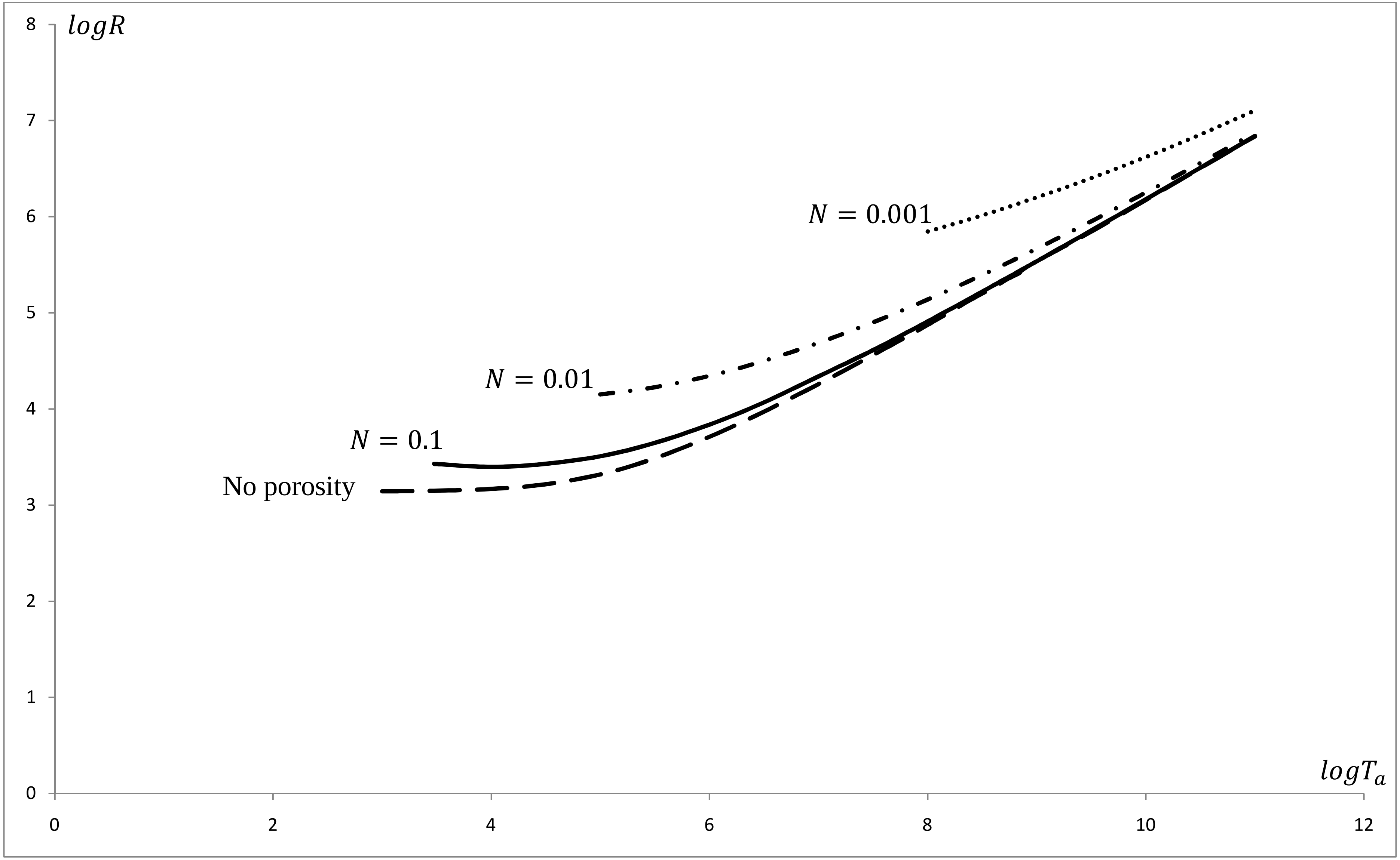

Figure 4. The relation between T_a and R for the overstability case when both boundaries are free for different values of N. Here $P_r = 0.05$.

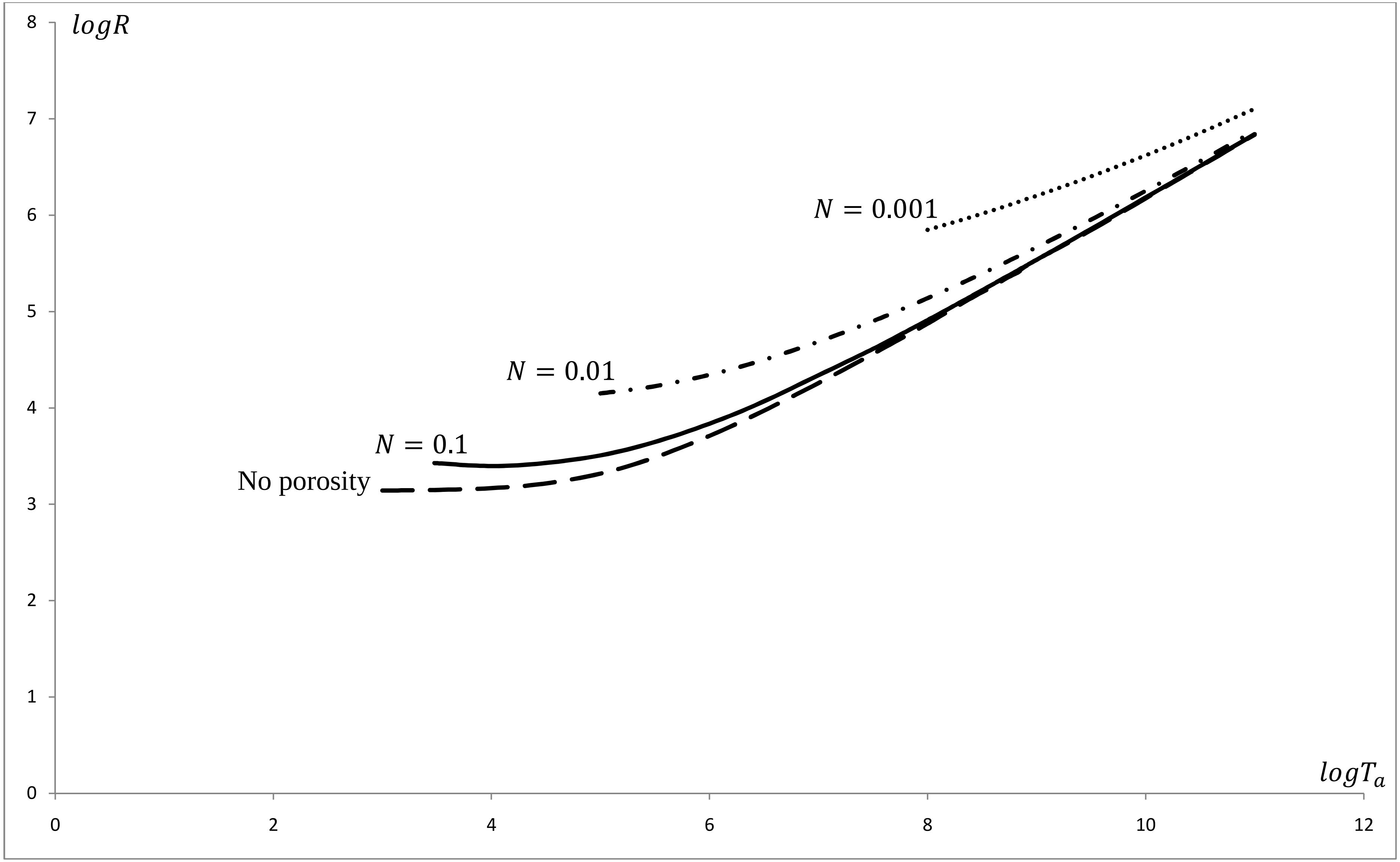

Figure 5. The relation between T_a and R for the overstability case when both boundaries are free for different values of N. Here $P_r = 0.025$.

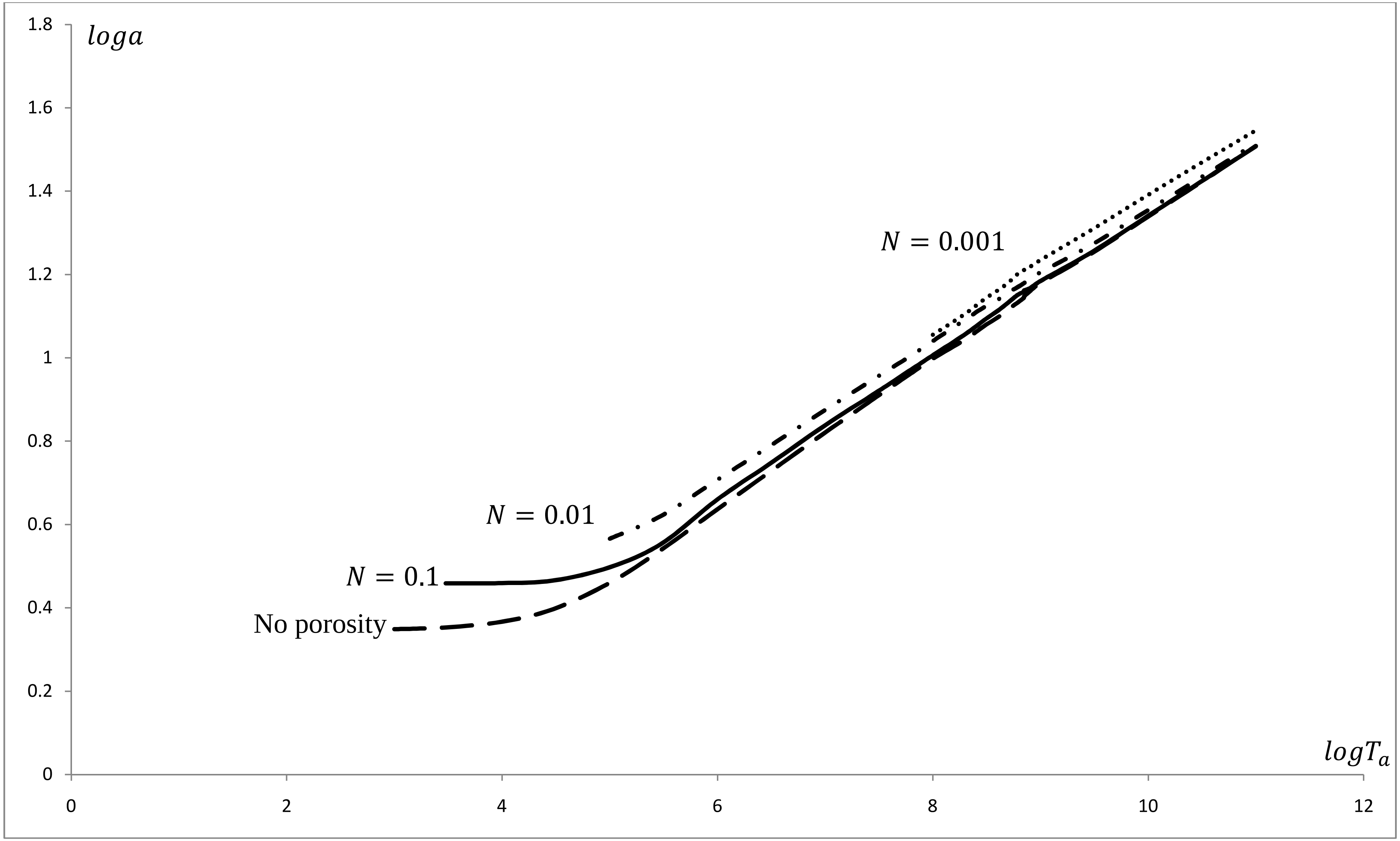

Figure 6. The relation between T_a and a for the overstability case when both boundaries are free for different values of N. Here $P_r = 0.05$

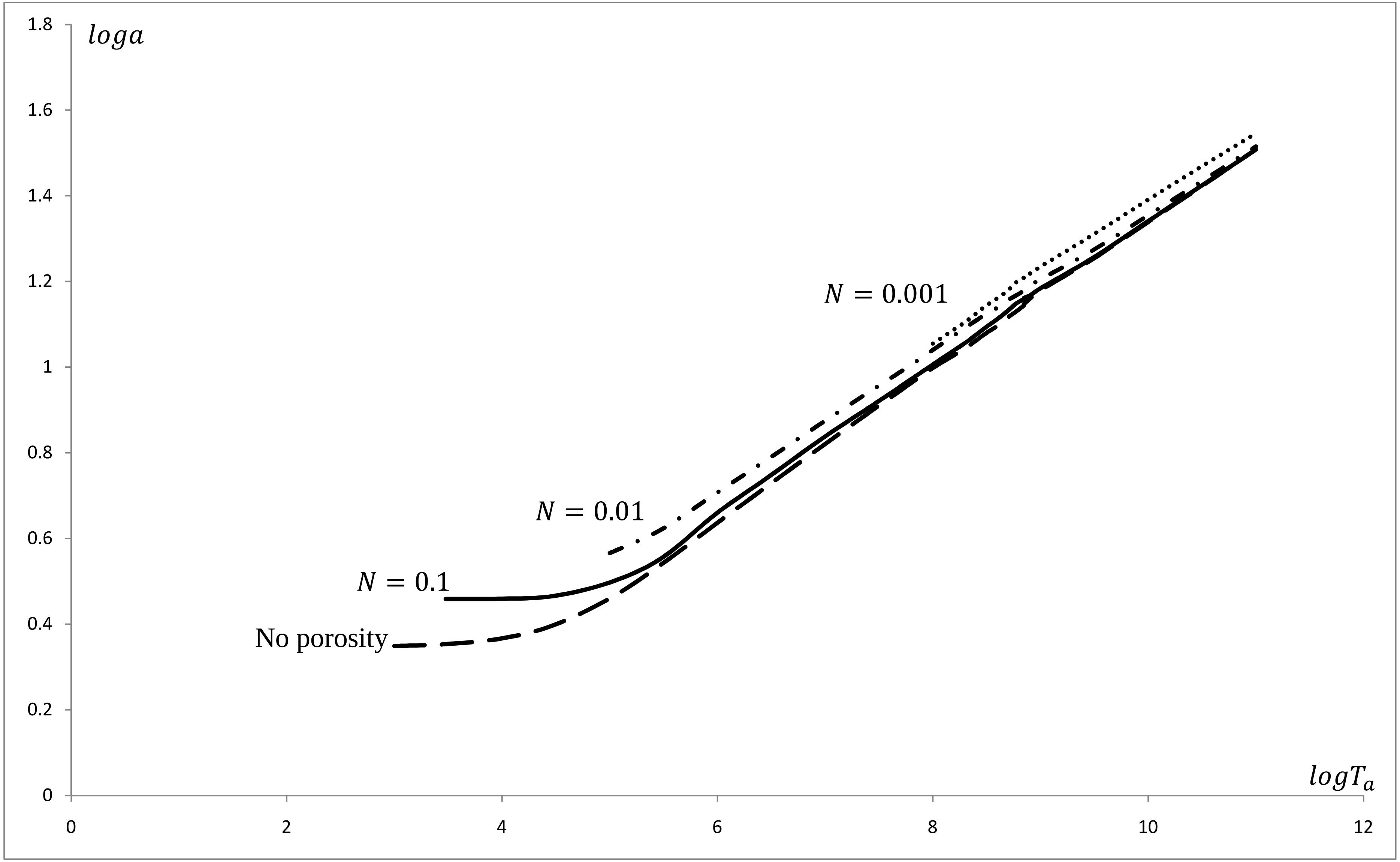

Figure 7. The relation between T_a and a for the overstability case when both boundaries are free for different values of N. Here $P_r = 0.025$.

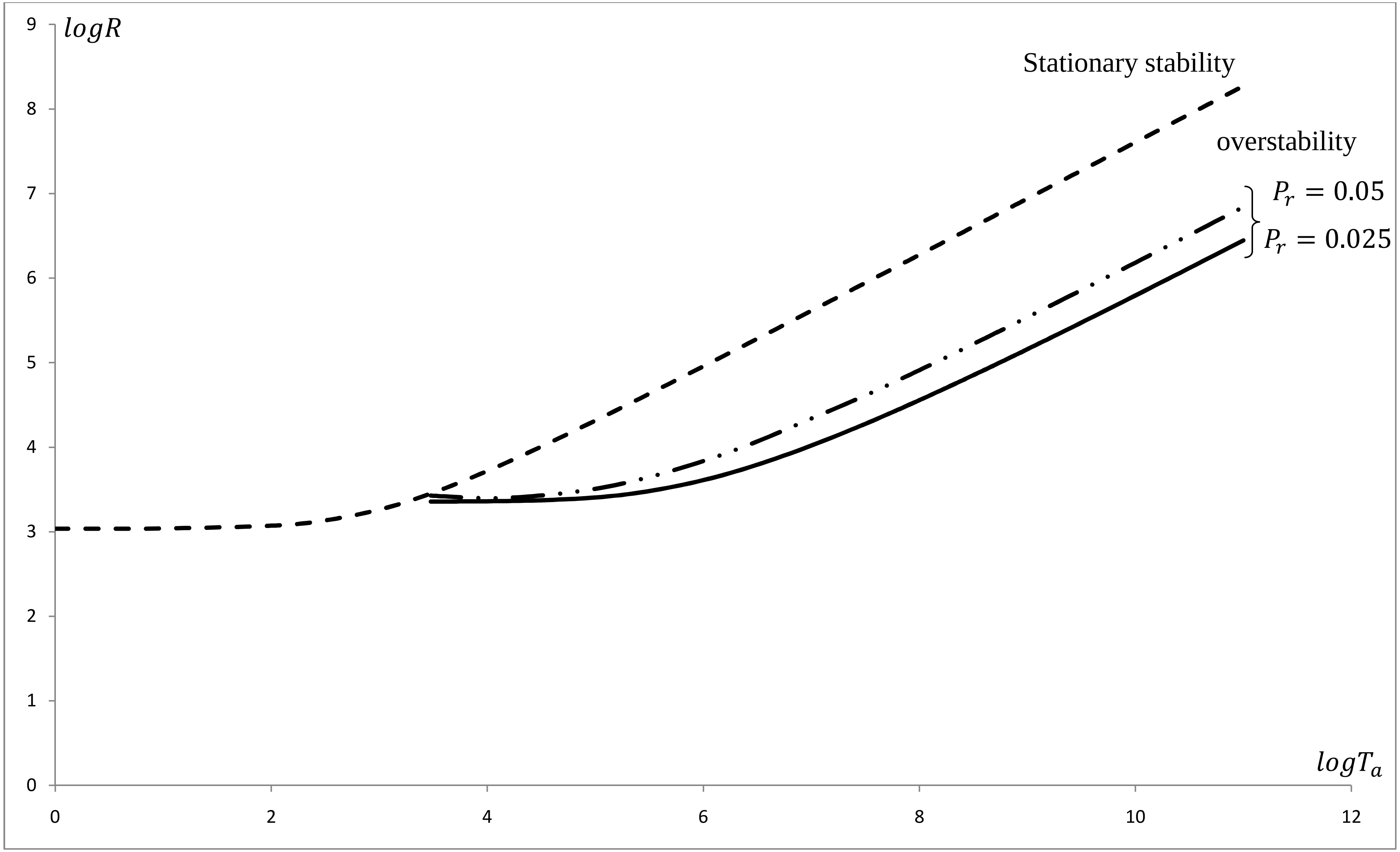

Figure 8. A comparison between the stationary and overstability cases when both boundaries are free for $N = 0.1$.

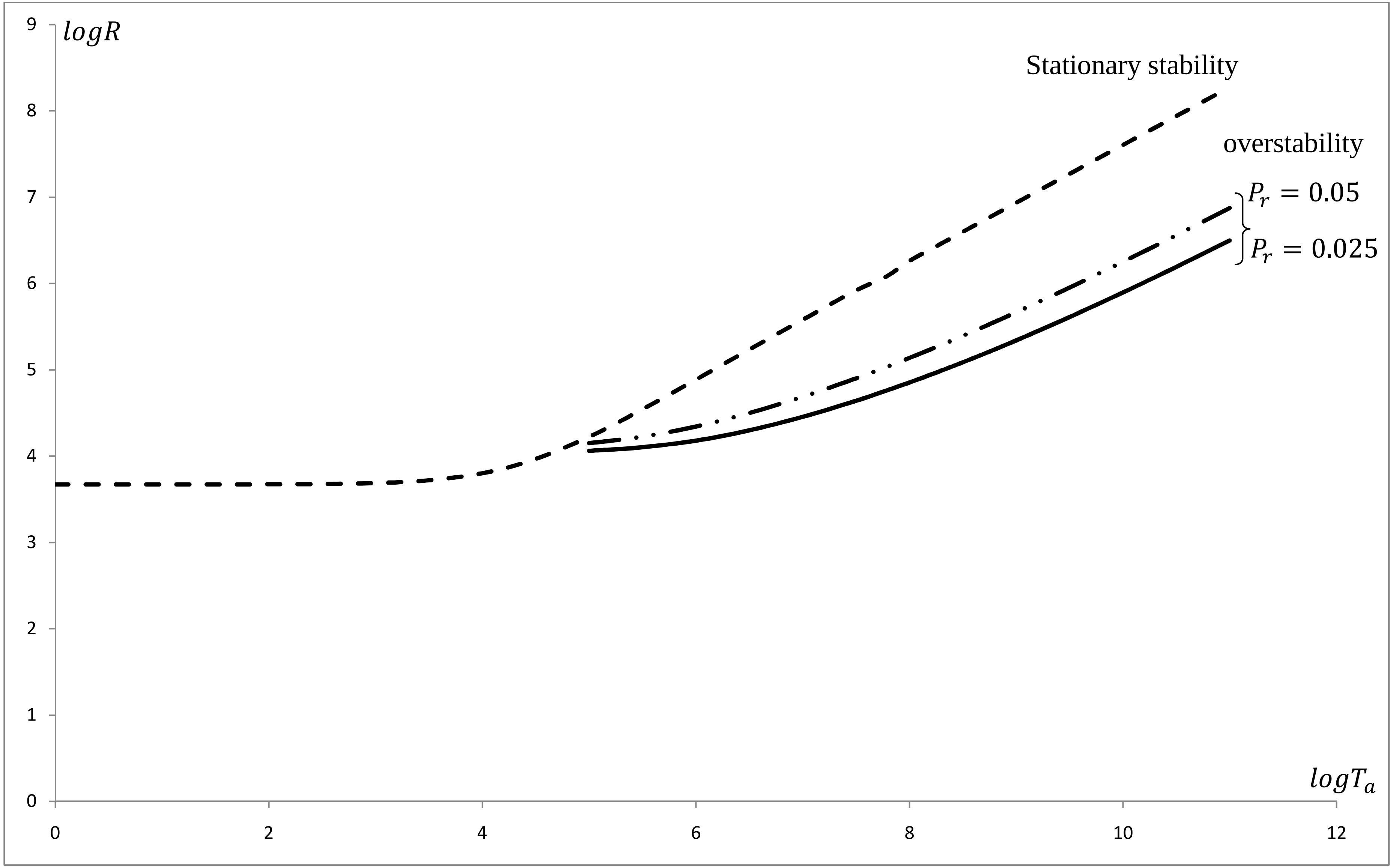

Figure 9. A comparison between the stationary and overstability cases when both boundaries are free for $N = 0.01$.

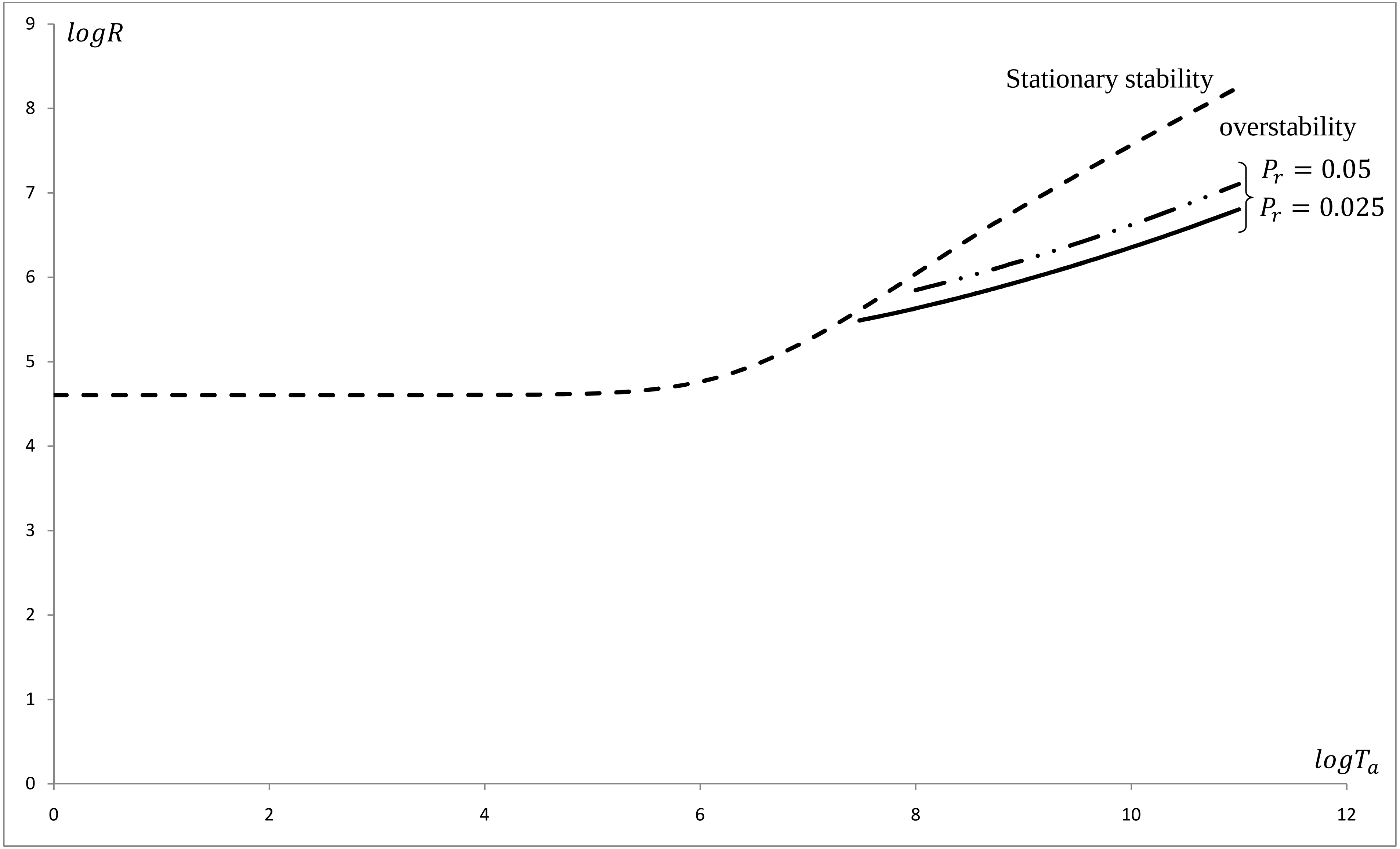

Figure 10. The comparison between the stationary and overstability cases when both boundaries are free for $N = 0.001$.

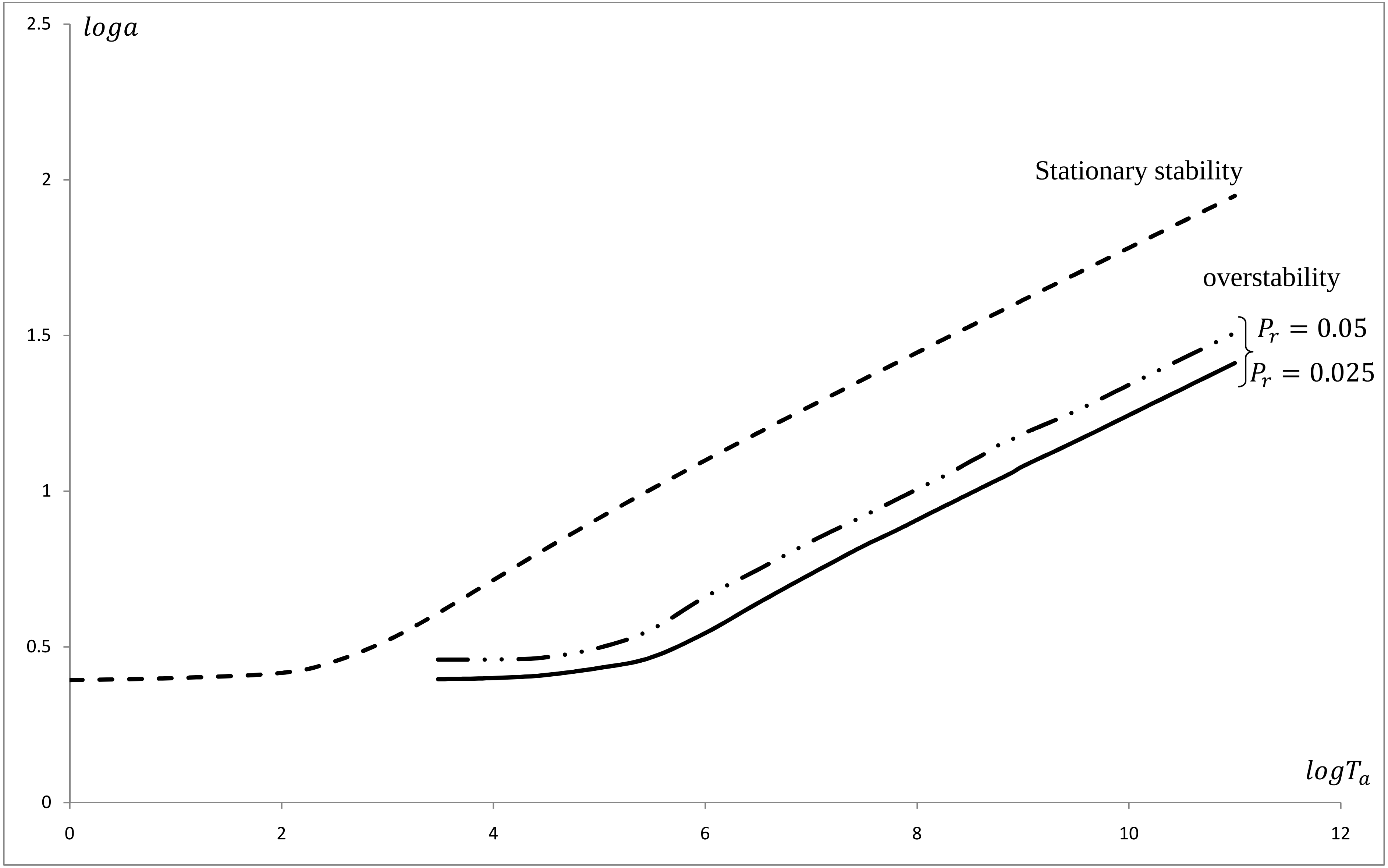

Figure 11. A comparison between the stationary and overstability cases when both boundaries are free for $N = 0.1$.

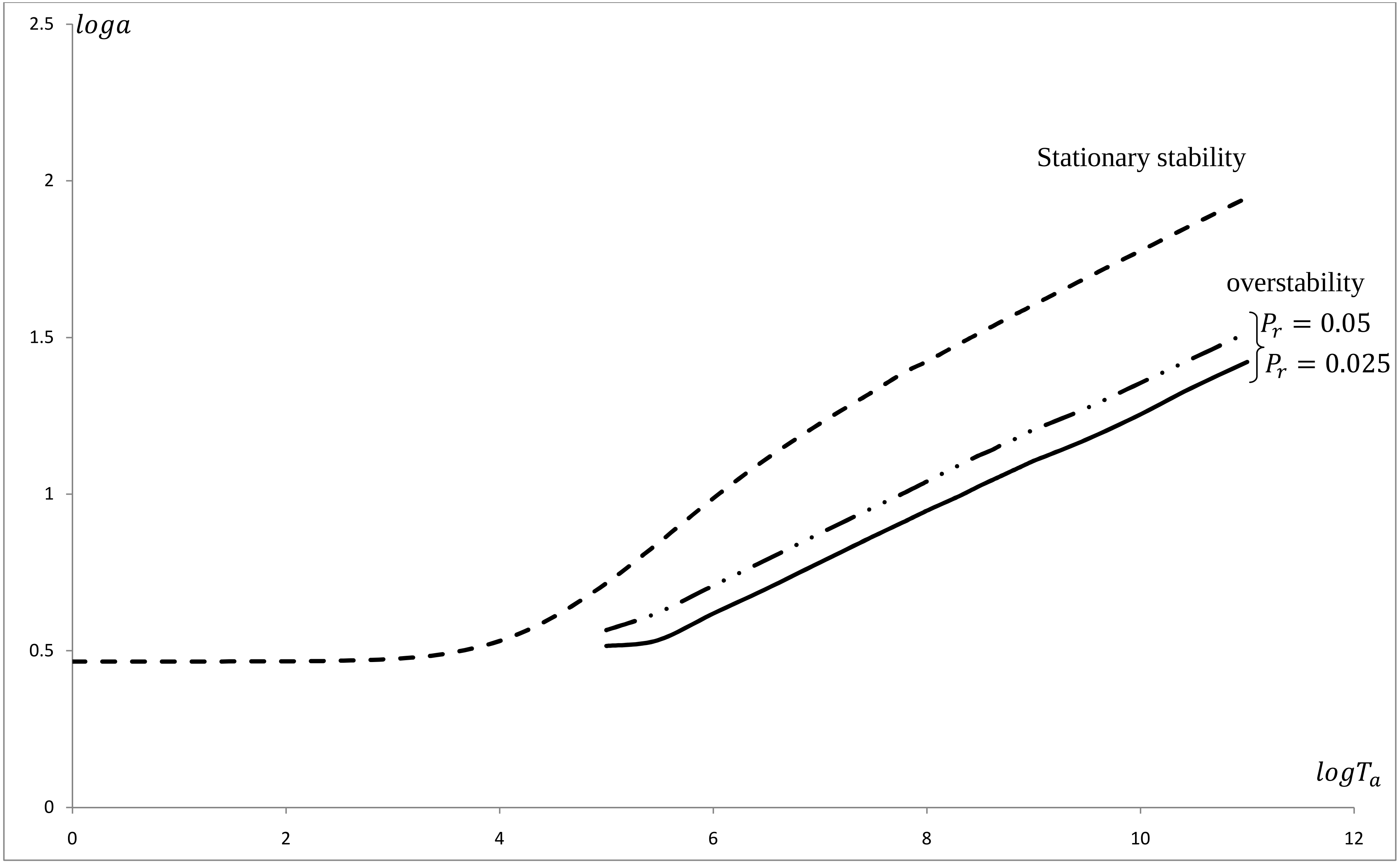

Figure 12. A comparison between the stationary and overstability cases when both boundaries are free for $N = 0.01$.

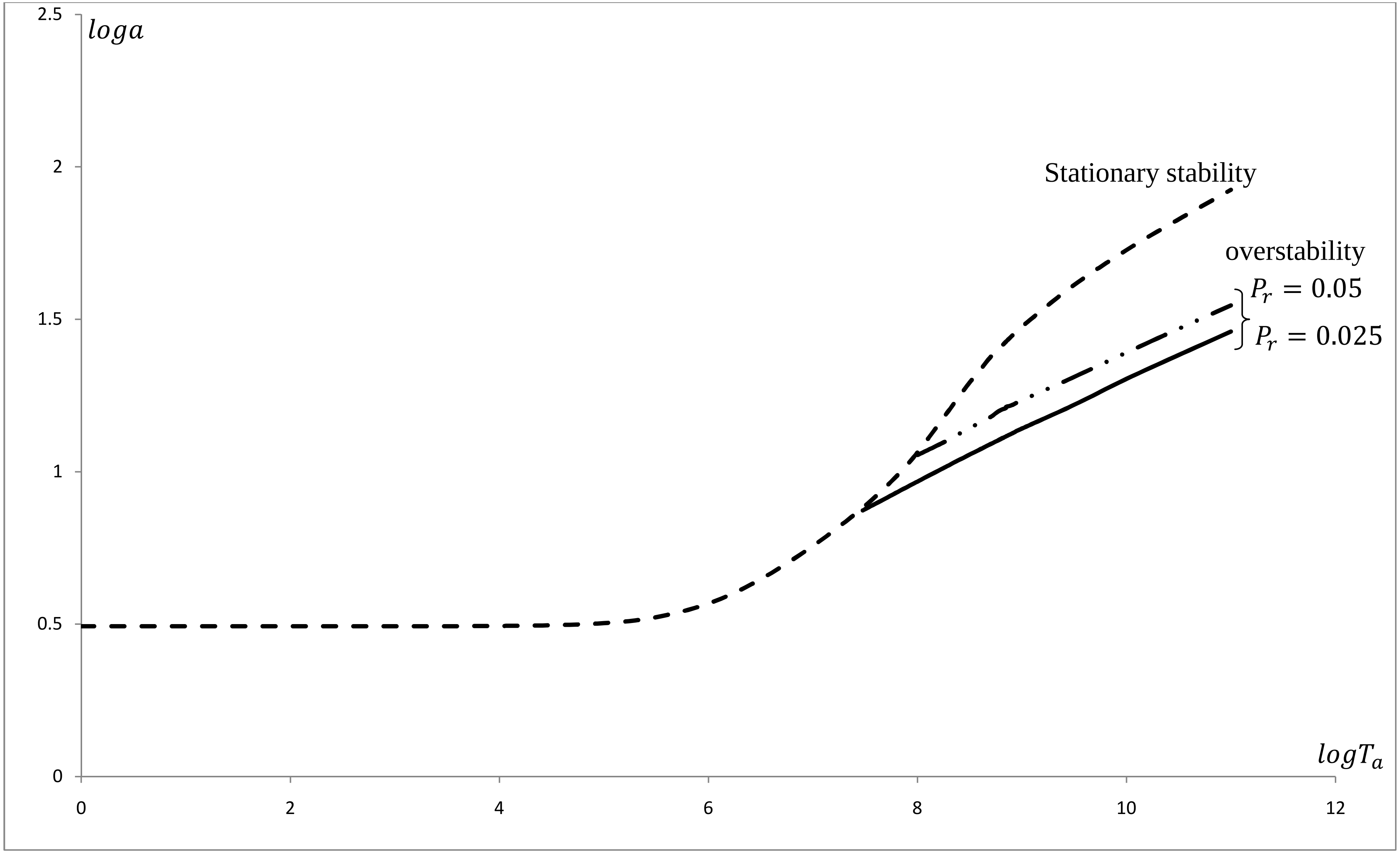

Figure 13. A comparison between the stationary and overstability cases when both boundaries are free for $N = 0.001$.

Table 4. The relation between T_a and R for the stationary stability case when both boundaries are rigid for different values of N.

T_a	NO POROUSITY		N=0.01		N=0.001		N=0.0001	
	a	R	a	R	a	R	a	R
0	3.121	1707.833	3.143	6043.749	3.167	43534.928	3.146	403102.040
10	3.131	1707.618	3.146	6046.138	3.167	43158.859	3.146	403832.846
100	3.146	1756.404	3.149	6058.364	3.181	43169.464	3.146	403834.638
300	3.167	1851.724	3.693	6202.627	3.236	43156.657	3.147	403895.626
1000	3.842	2145.023	4.573	7055.330	3.511	43509.903	3.147	403834.768
3000	4.384	2912.103	5.487	8294.560	4.056	45581.520	3.170	403845.960
10000	5.191	4754.954	6.573	11638.425	4.780	47918.118	3.328	410926.868
30000	6.382	8471.805	8.147	18525.312	5.618	59002.716	3.573	409671.810
100000	7.808	16952.395	9.853	29985.803	6.950	77107.569	4.309	443296.222
1000000	11.475	71161.884	14.331	99881.784	10.528	158489.169	6.286	626668.808
10000000	18.978	337904.200	21.477	461552.508	17.7523	490407.530	13.119	1974007.042
20000000	22.013	557599.459	25.283	737122.573	21.2812	821426.638	17.646	3181672.117
30000000	24.235	720945.890	27.717	981556.565	23.854	1107646.298	20.223	4502838.908
60500000	28.175	1292378.856	31.2527	1553012.211	27.539	1689568.473	25.390	7747028.431
100000000	31.581	1919024.767	35.413	2429392.840	31.282	2441544.220	31.123	10947502.774
200000000	36.827	3080868.716	40.369	3855911.694	36.673	3829150.542	37.223	16192482.332
300000000	39.768	3965421.657	44.413	4912892.201	41.1734	5636235.058	40.270	20566293.824
405000000	43.579	5553119.440	47.0902	6717770.617	43.5042	6800674.935	43.1672	24604653.55
500000000	45.918	6571371.922	48.1459	7458010.811	45.7627	8004851.943	45.7309	25955585.09

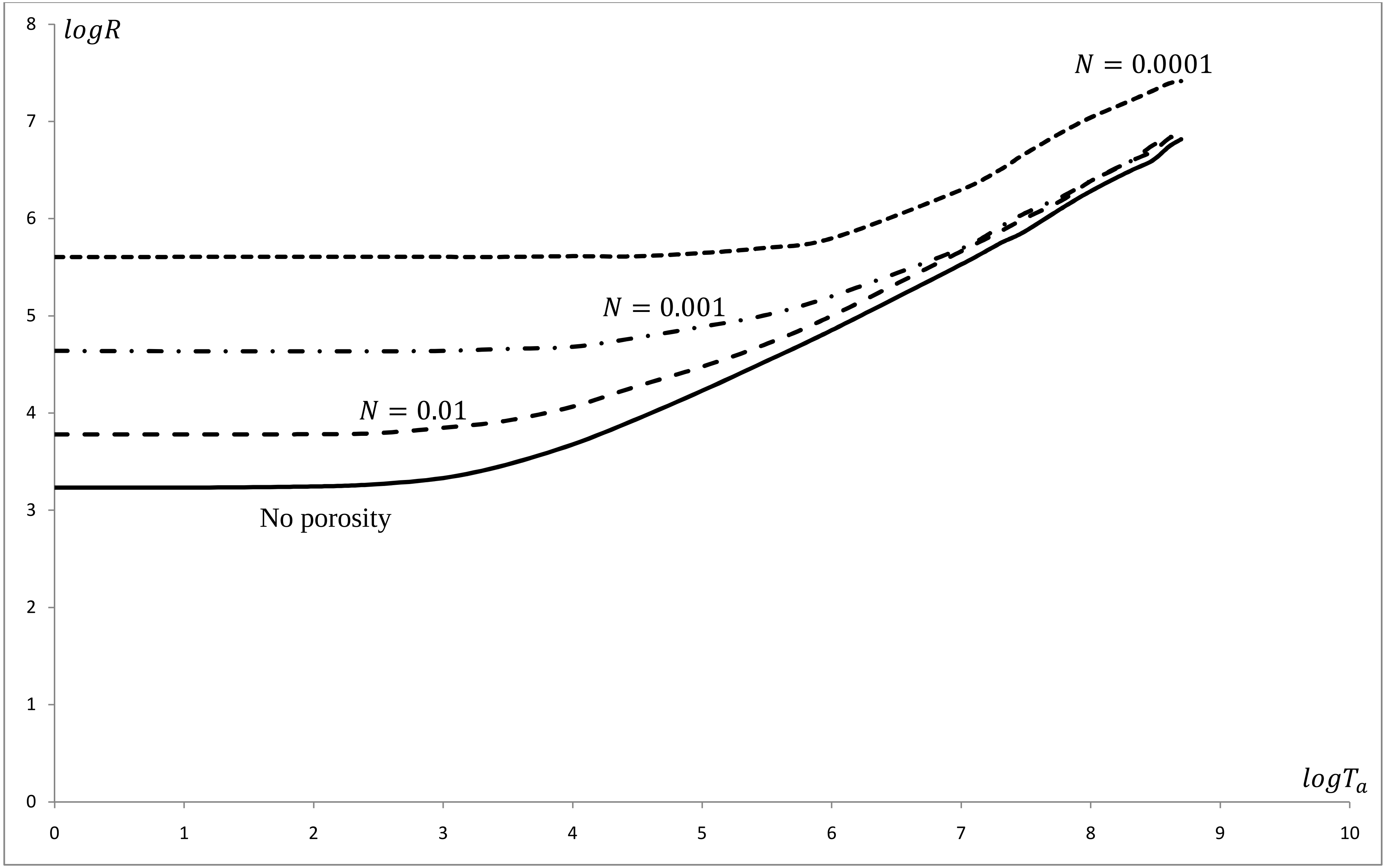

Figure14. The relation between T_a and R for the stationary stability case when both boundaries are rigid for different values of N.

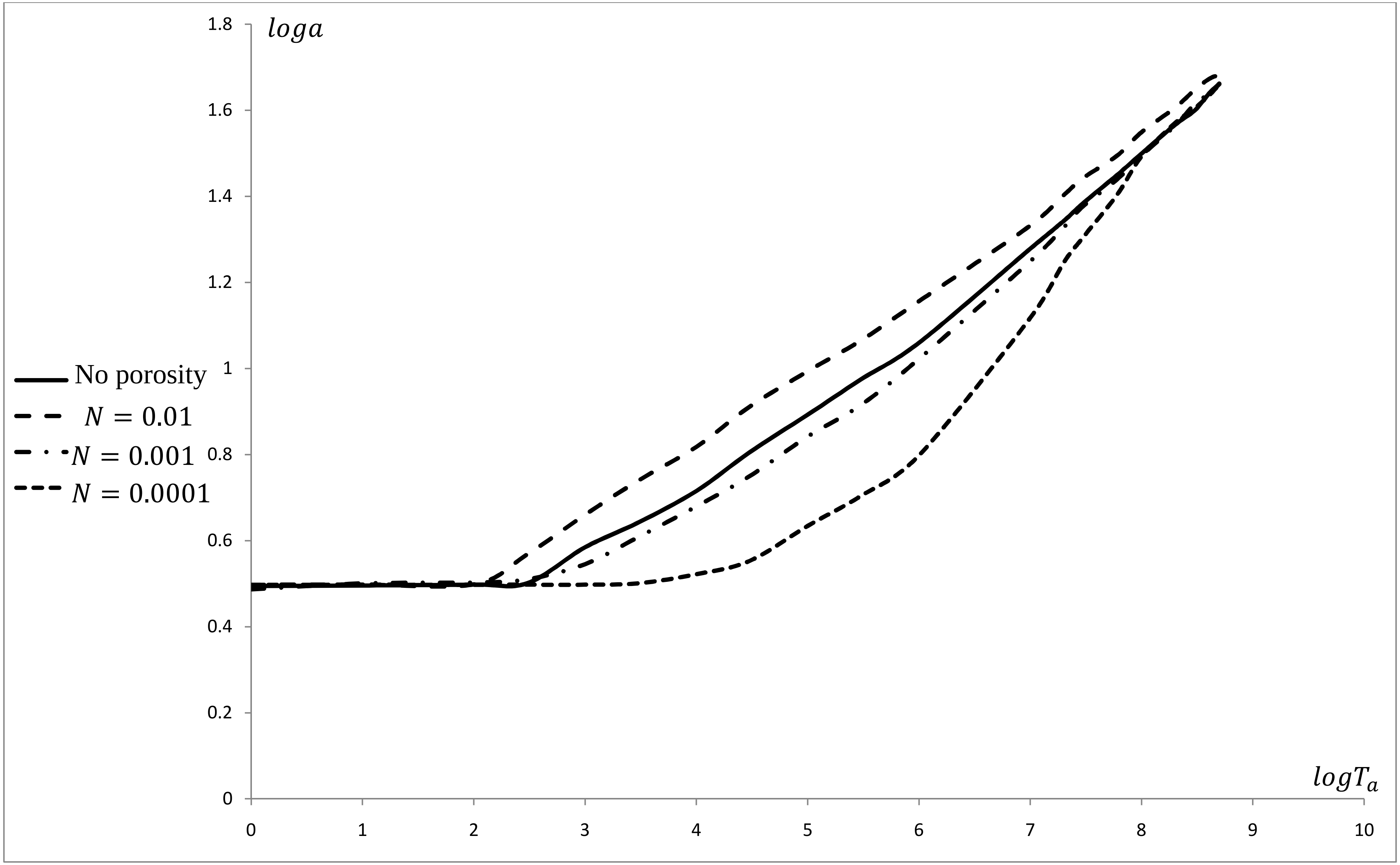

Figure 15. The relation between T_a and a for the stationary stability case when both boundaries are rigid for different values of N.

Table 5. The relation between T_a and R for the overstability case when both boundaries are rigid for different values of N and $P_r = 0.05$.

T_a	NO POROUSITY		$N=0.01$		$N=0.001$	
	a	R	a	R	a	R
0						
300000	4.191	9233.491				
1000000	5.146	15038.833	7.482	43000.879		
10000000	7.854	47859.980	8.717	83337.675		
20000000	8.613	69869.493	9.294	110410.074		
30000000	9.192	87772.165	9.754	132953.796		
50000000	10.019	117509.261	10.236	168683.389		
60500000	10.347	131169.110	10.564	185306.403		
70000000	10.604	142736.043	10.821	199274.055		
80500000	10.856	154819.769	11.074	213772.698		
90500000	11.072	165761.445	11.289	226827.601	11.305	821689.410
100000000	11.258	175722.482	11.477	238657.640	11.490	849269.071
200000000	12.633	264366.622	12.860	342174.789	12.997	1079160.819
205000000	12.694	268265.299	12.912	346670.636	13.054	1088754.286
300000000	13.524	336435.497	13.741	424695.438	13.967	1251203.692
405000000	14.215	402461.782	14.392	499439.790	14.700	1401278.583
500000000	14.920	456570.908	14.828	562068.085	15.288	1521398.113
605000000	15.678	511894.253	15.779	623479.158	15.750	1636926.756
700000000	16.326	561973.736	16.483	679738.775	16.147	1738339.019
705000000	16.567	570818.139	16.764	687325.237	16.166	1751468.884
805000000	17.208	624557.508	16.818	731072.908	16.535	1841443.608
905000000	17.878	666136.409	17.541	791736.087	16.866	1924576.772

Table 6. The relation between T_a and R for the overstability case when both boundaries are rigid for different values of N. Here $P_r = 0.025$.

T_a	NO POROUSITY		N=0.01		N=0.001	
	a	R	a	R	a	R
0						
10000	3.087	4379.261				
30000	3.121	4640.439				
100000	3.122	5217.530	4.157	17383.486		
300000	3.451	6491.043	4.221	19111.116		
1000000	4.092	9509.511	4.695	23603.599		
10000000	6.110	26757.652	6.317	48186.584		
30000000	7.410	47640.985	7.608	75956.685	7.576	321987.977
60500000	8.368	70025.209	8.563	104377.257	8.584	440516.352
70000000	8.581	75963.671	8.775	111766.655	8.810	460129.634
80500000	8.789	82157.225	8.983	119420.757	9.032	480138.208
100000000	9.122	92848.805	9.316	132522.797	9.388	513720.965
200000000	10.264	137943.825	10.456	186635.326	10.612	645146.246
300000000	10.993	174395.033	11.183	229431.669	11.392	742762.013
405000000	11.562	207671.479	11.752	267994.222	12.011	827163.619
500000000	11.979	234871.365	12.168	299237.159	12.444	893544.082
605000000	12.368	262610.104	12.556	330888.623	12.858	959250.218
700000000	12.674	286079.728	12.861	357527.026	13.183	1013496.727
705000000	12.689	287278.473	12.876	358884.456	13.199	1016237.270
805000000	12.973	310585.162	13.160	385219.974	13.501	1068981.307
905000000	13.228	332770.189	13.415	410195.278	13.772	1118298.288

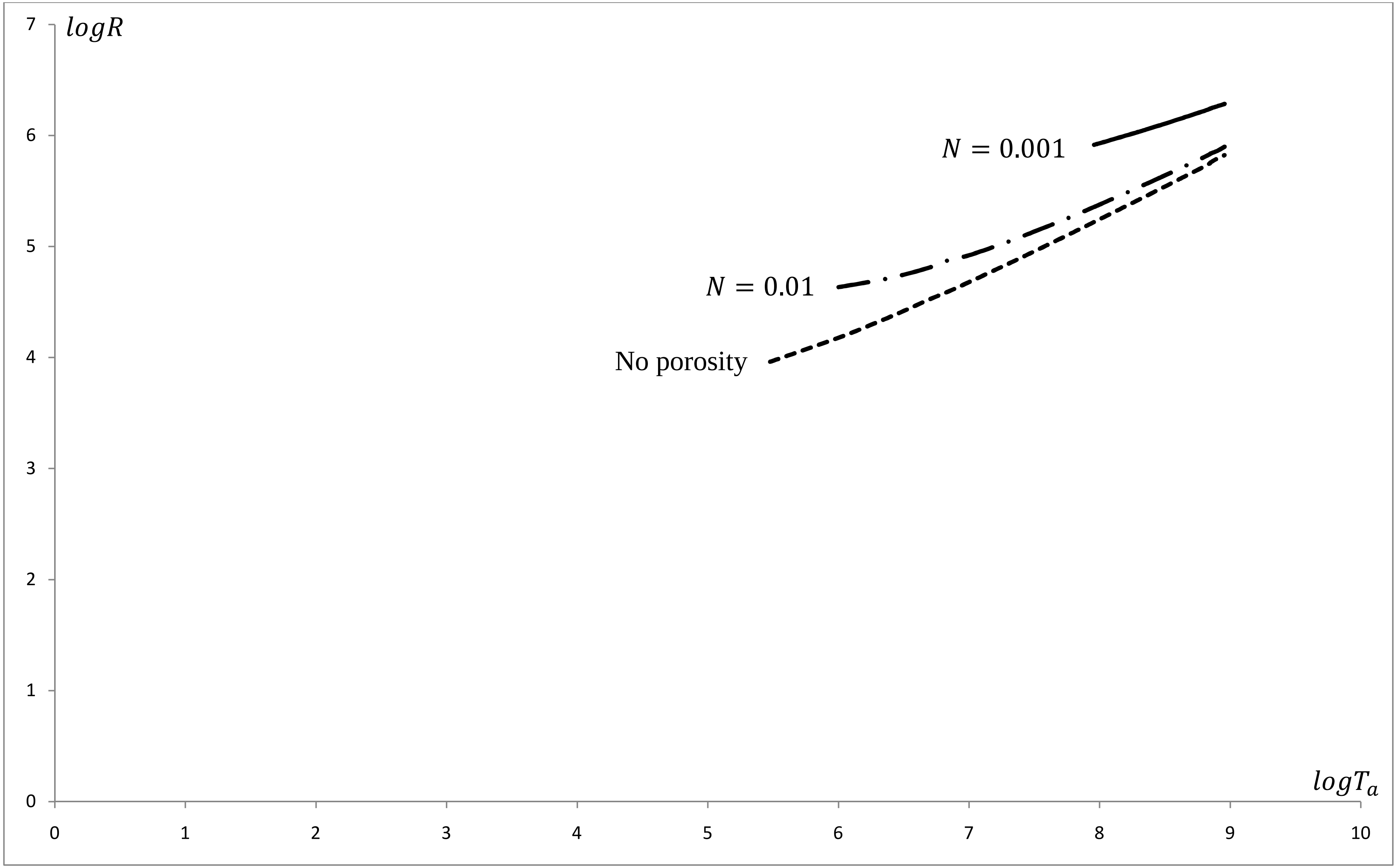

Figure16. The relation between T_a and R for the overstability case when both boundaries are rigid for different values of N. Here $P_r = 0.05$.

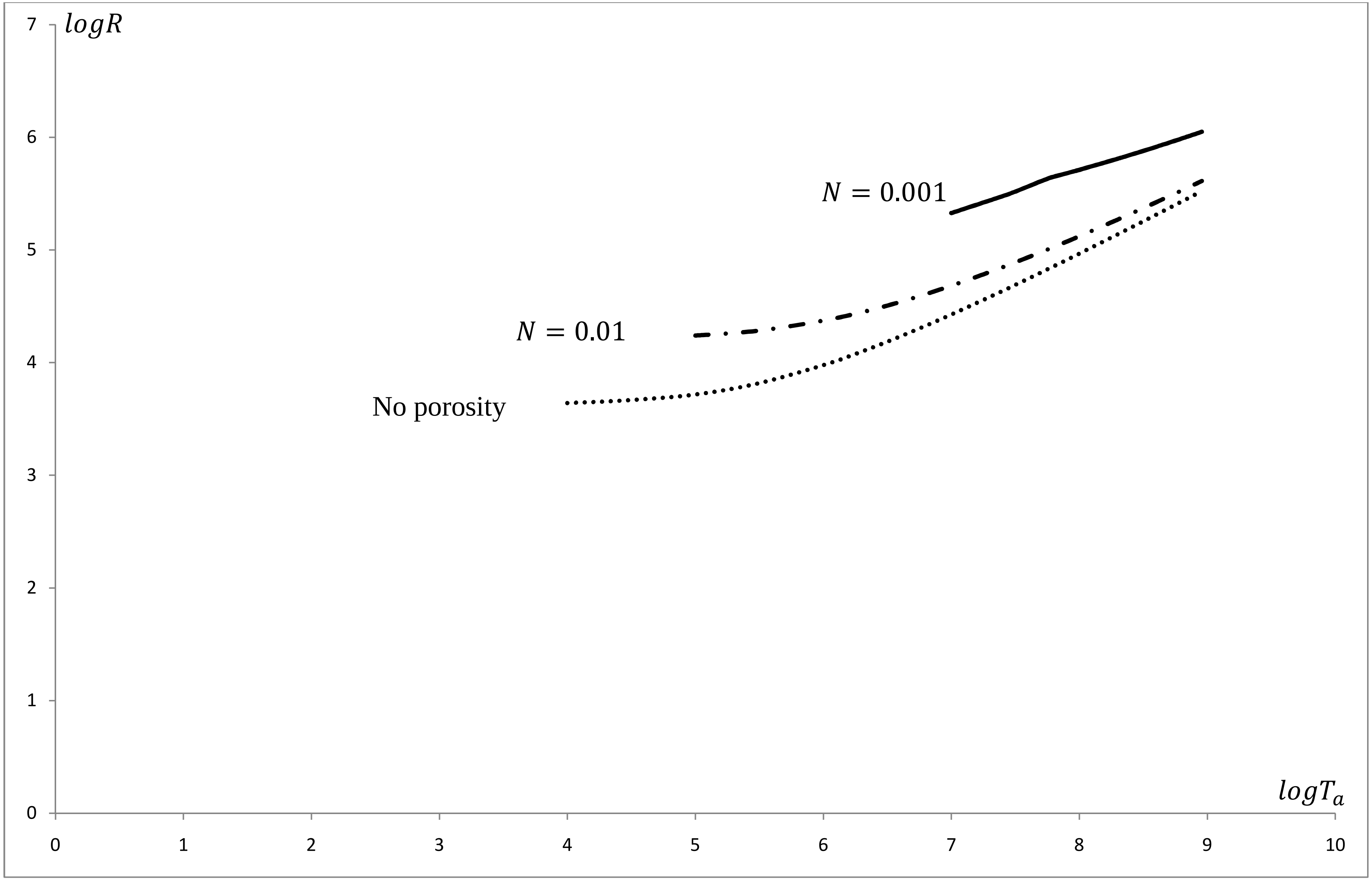

Figure17. The relation between T_a and R for the overstability case when both boundaries are rigid for different values of N. Here $P_r = 0.025$.

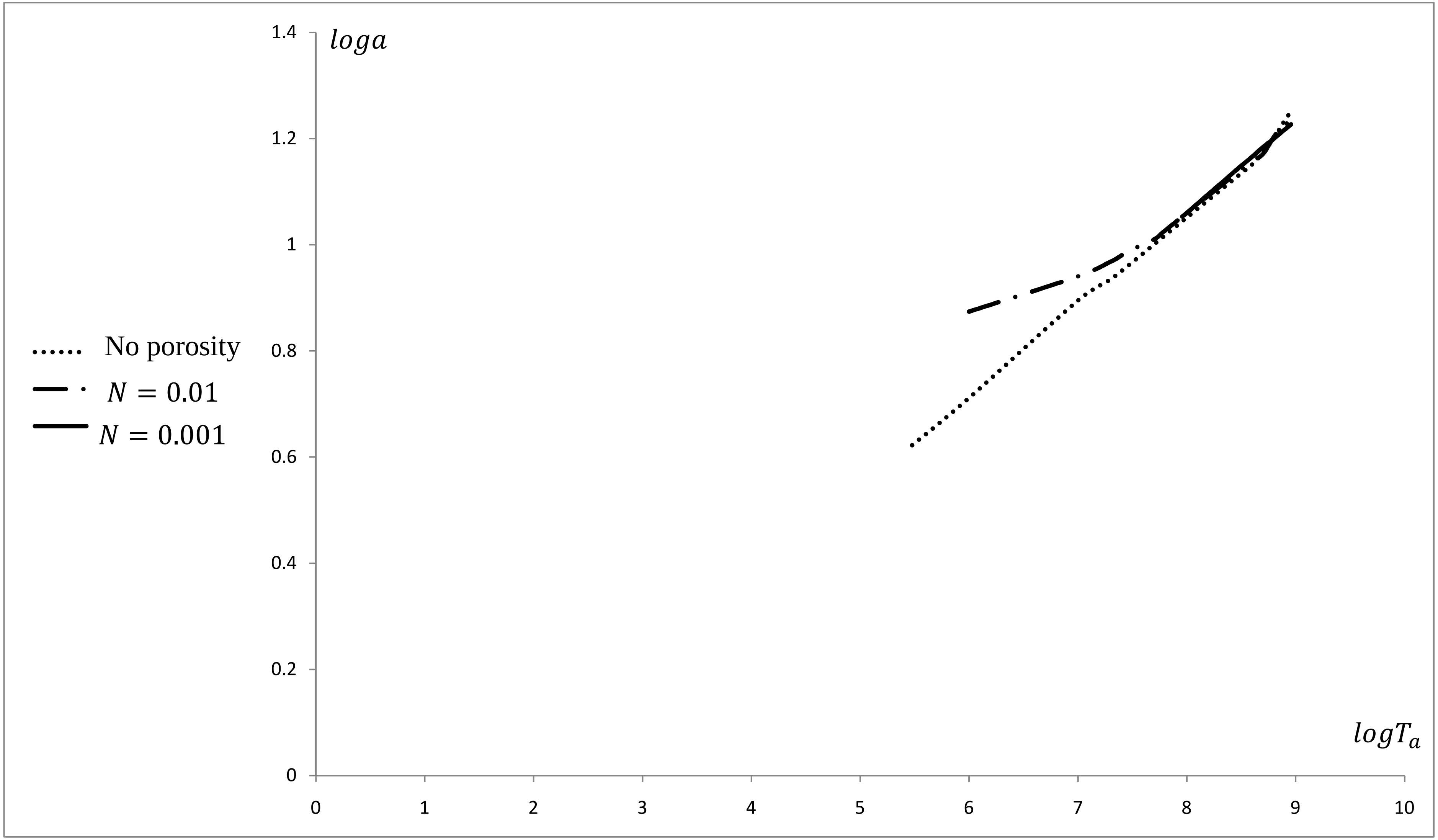

Figure18. The relation between T_a and a for the overstability case when both boundaries are rigid for different values of N. Here $P_r = 0.05$.

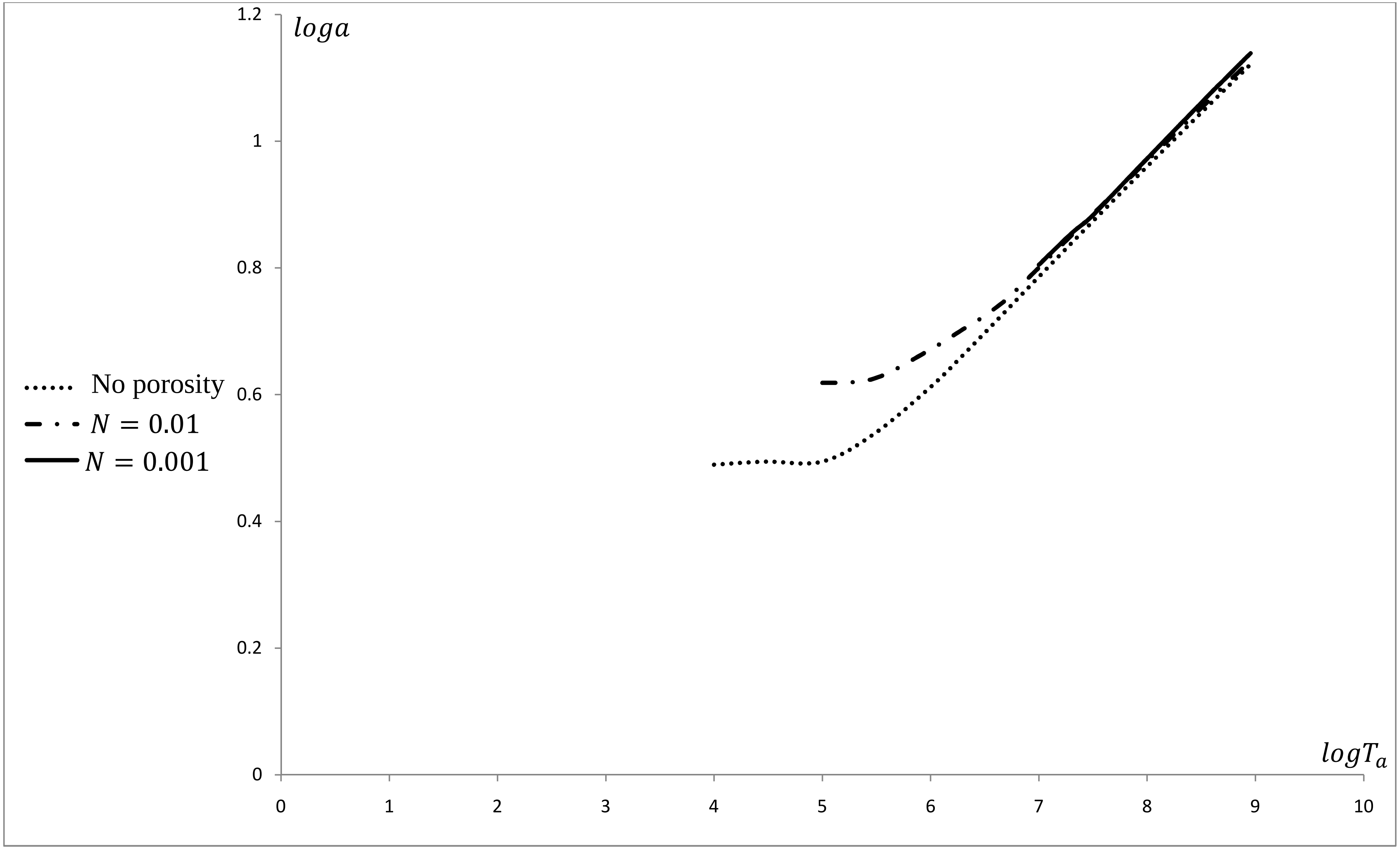

Figure 19. The relation between T_a and a for the overstability case when both boundaries are rigid for different values of N. Here $P_r = 0.025$.

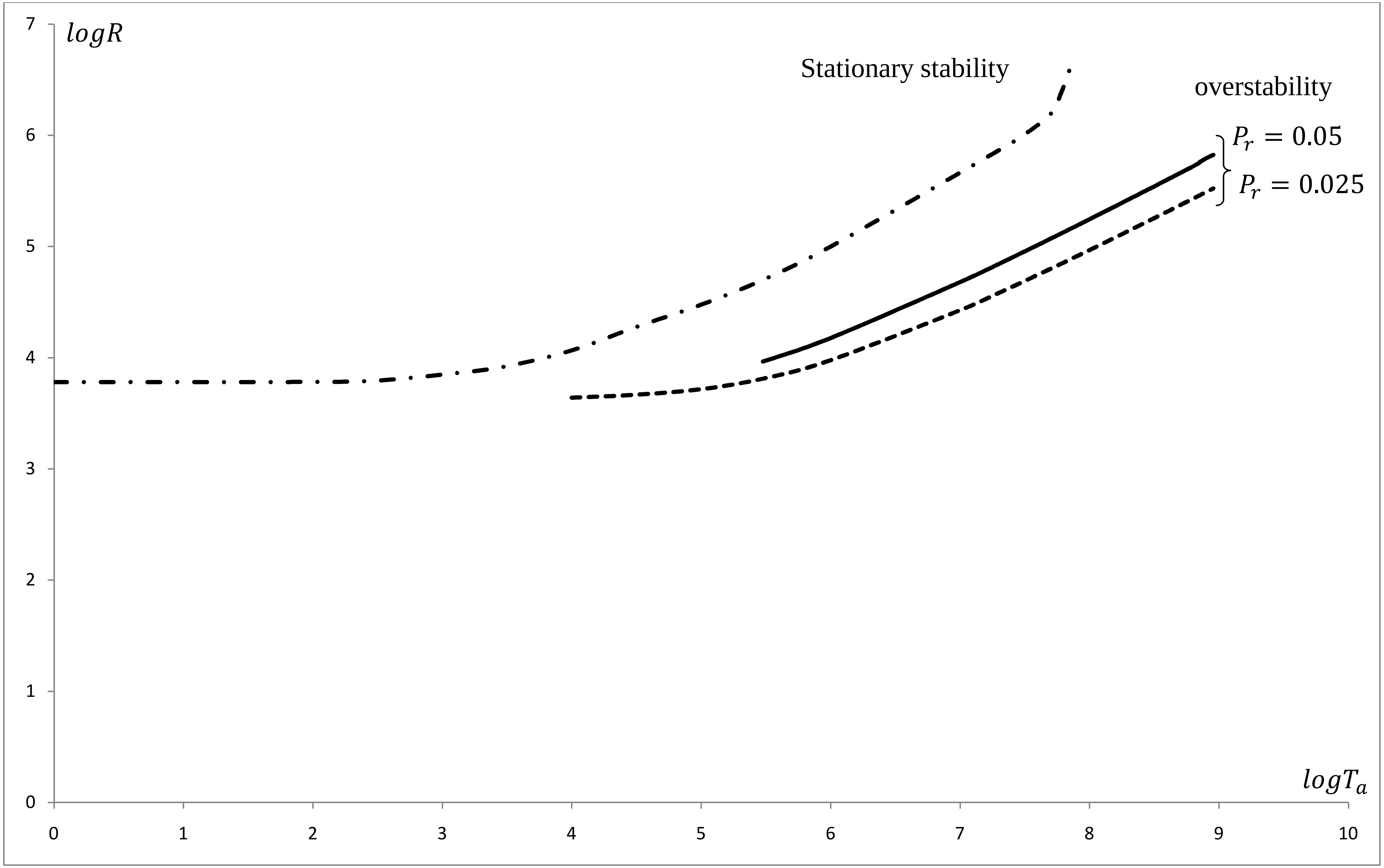

Figure 20. A comparison between the stationary and overstability cases when both boundaries are rigid for $N = 0.01$.

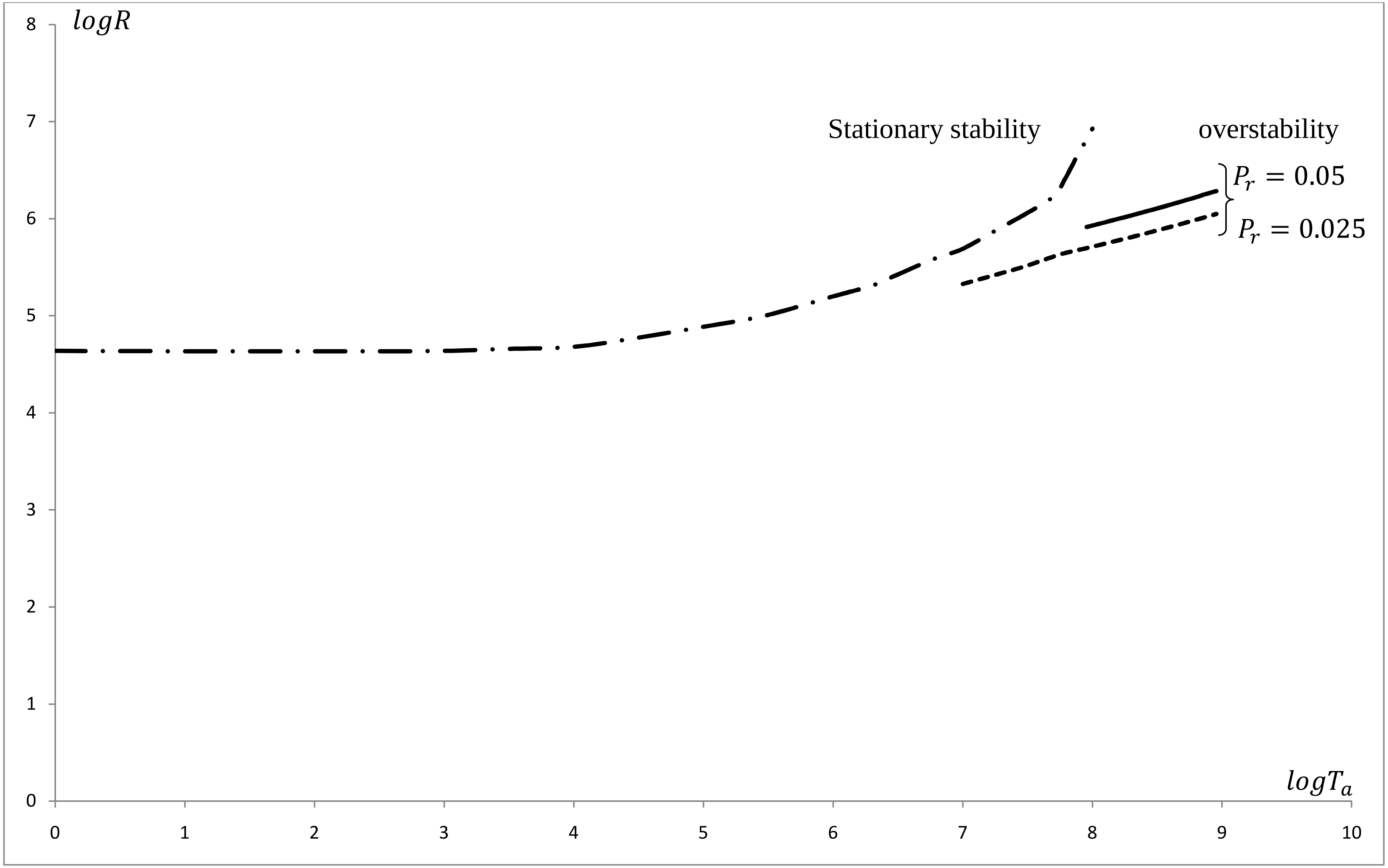

Figure 21. A comparison between the stationary and overstability cases when both boundaries are rigid for $N = 0.001$.

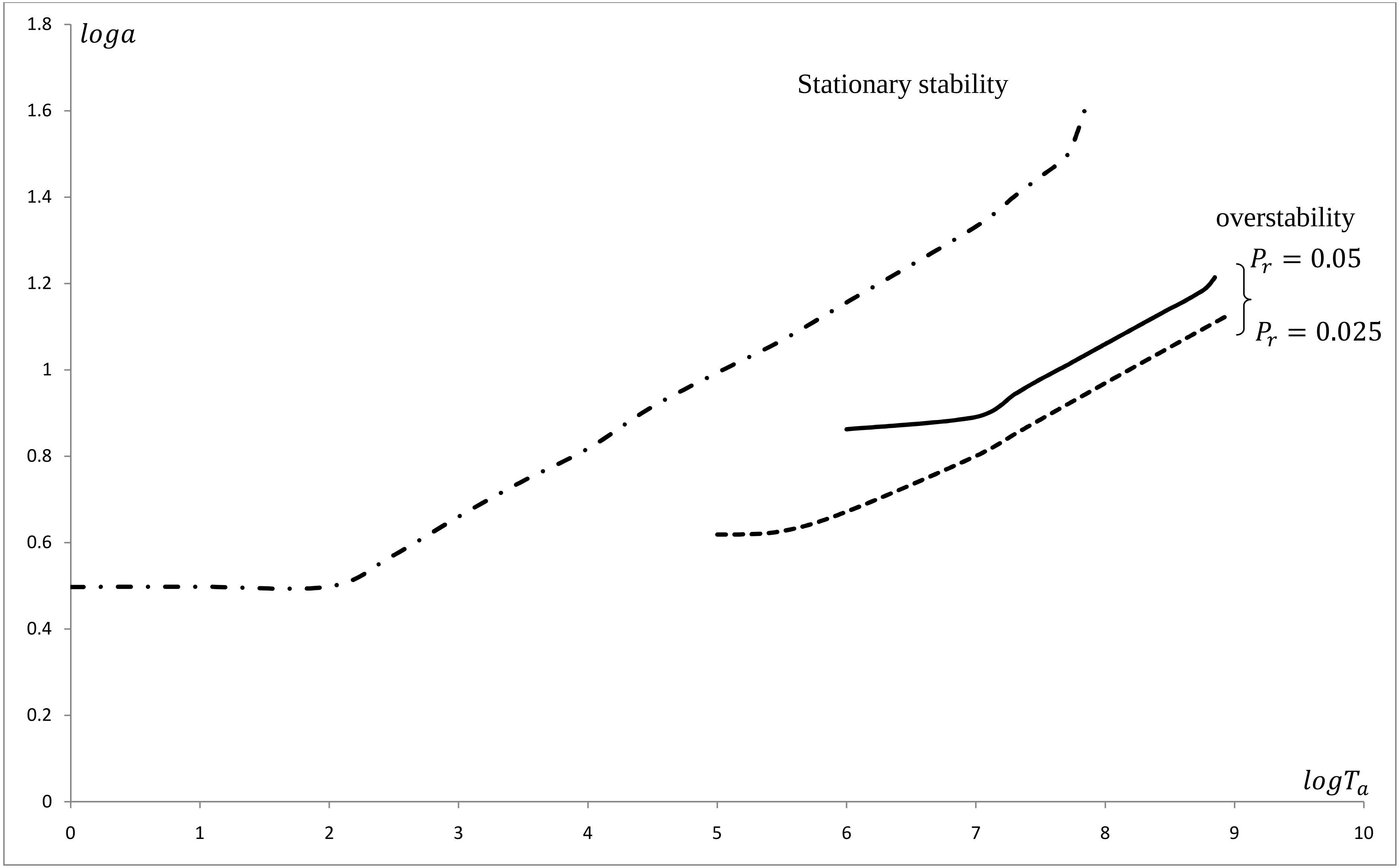

Figure 22. A comparison between the stationary and overstability cases when both boundaries are rigid for $N = 0.01$.

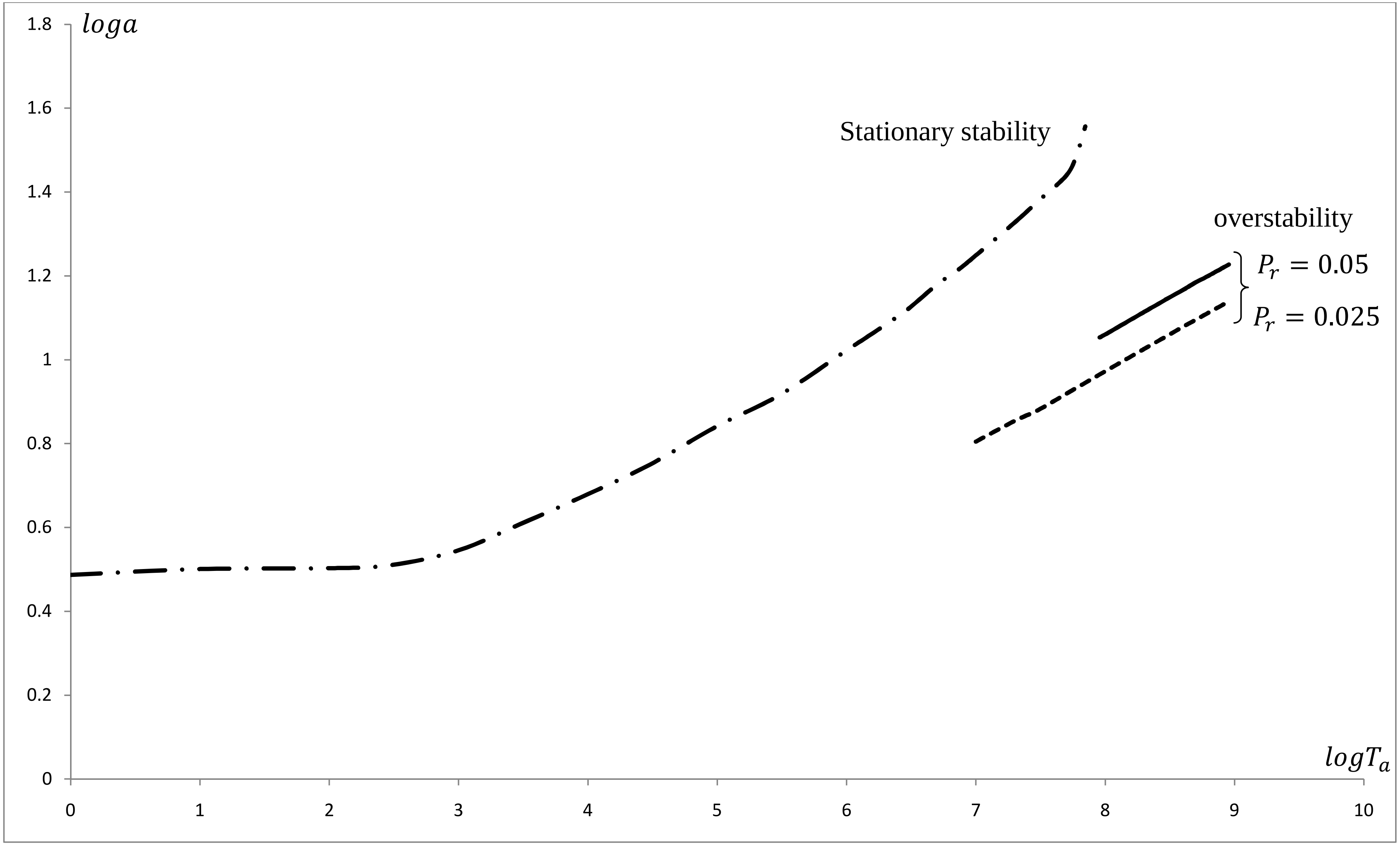

Figure 23. A comparison between the stationary and overstability cases when both boundaries are rigid for $N = 0.001$.

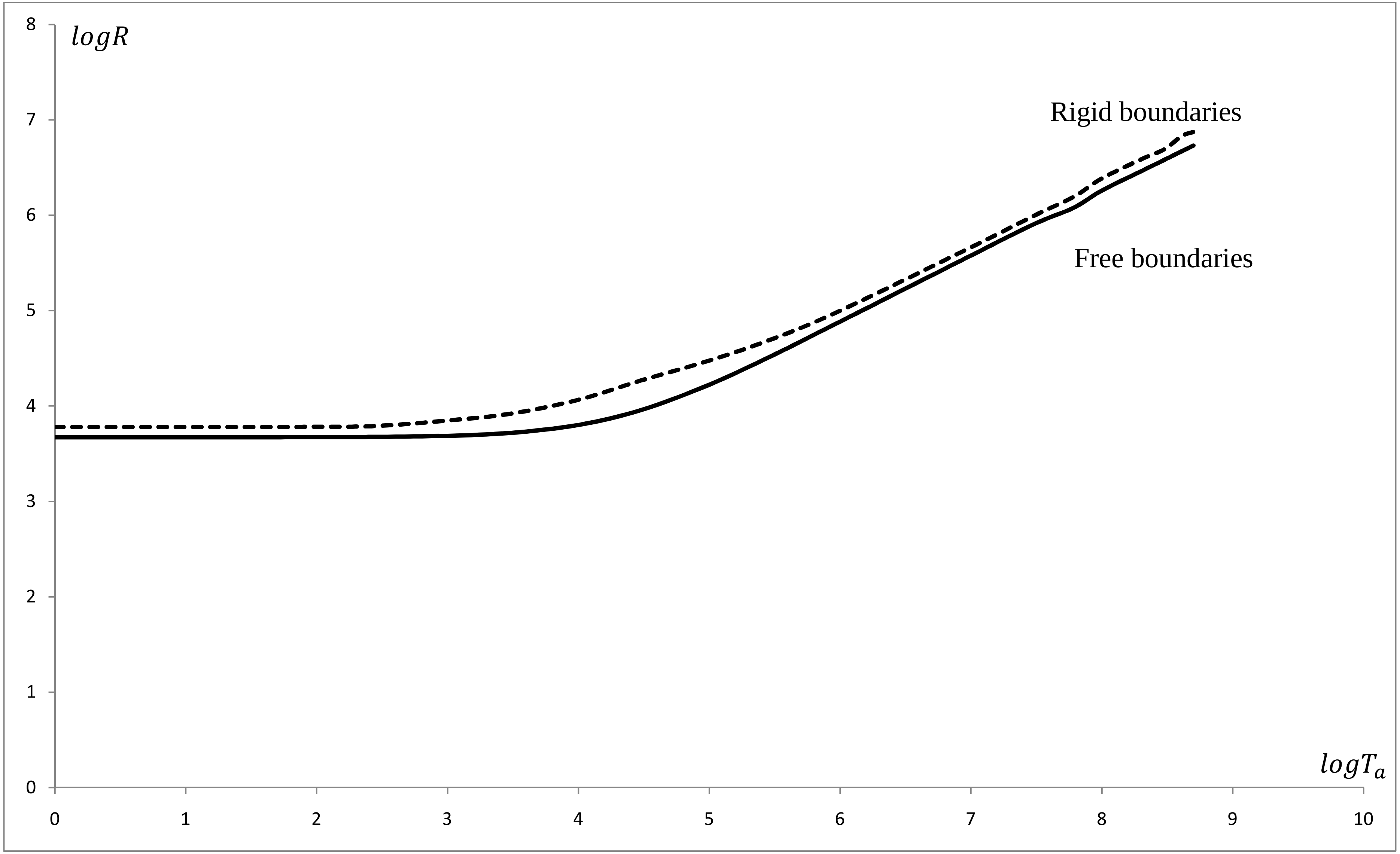

Figure 24. A comparison between free and rigid boundaries for the stationary stability case when $N = 0.01$.

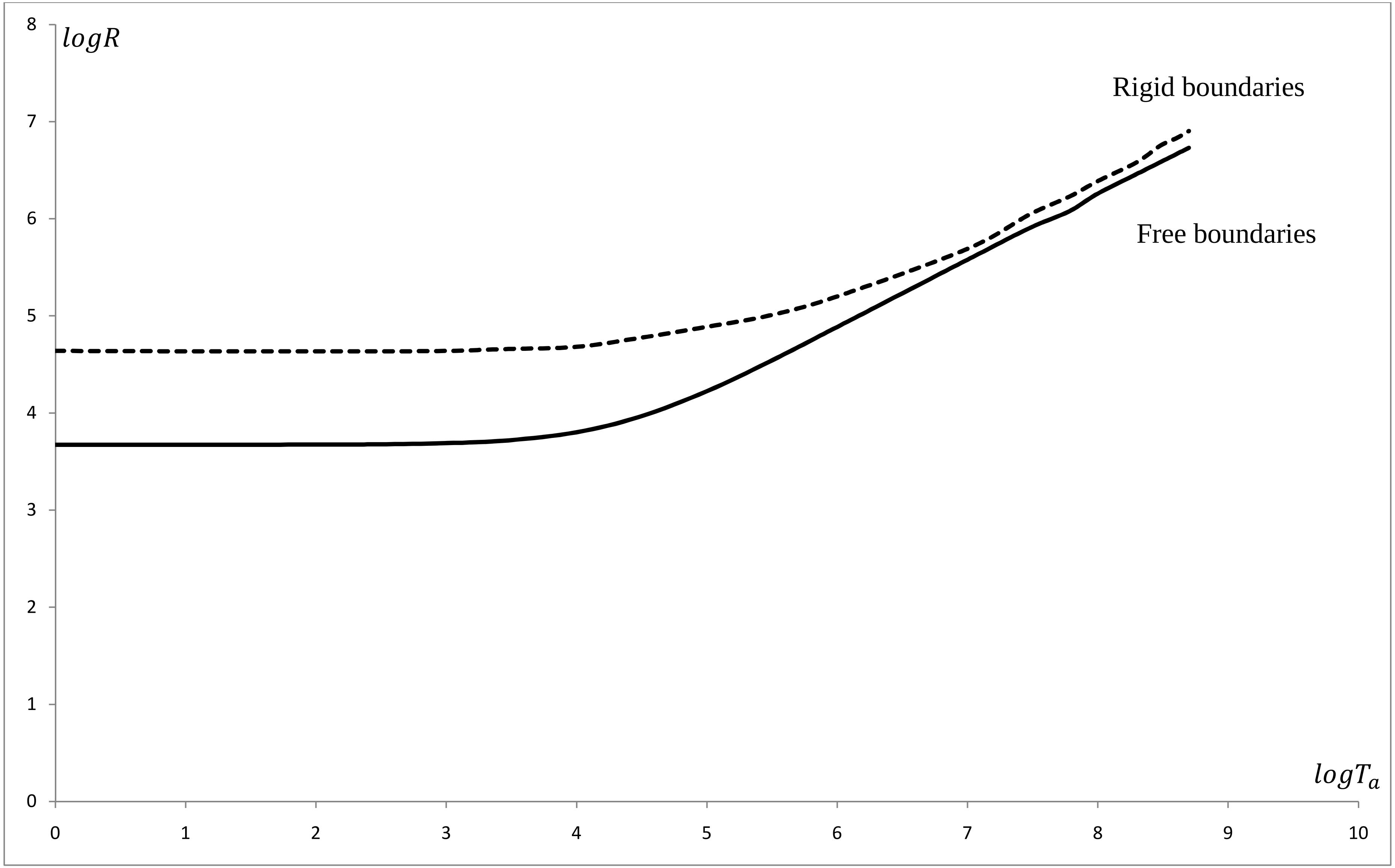

Figure 25. A comparison between free and rigid boundaries for the stationary stability case when $N = 0.001$.

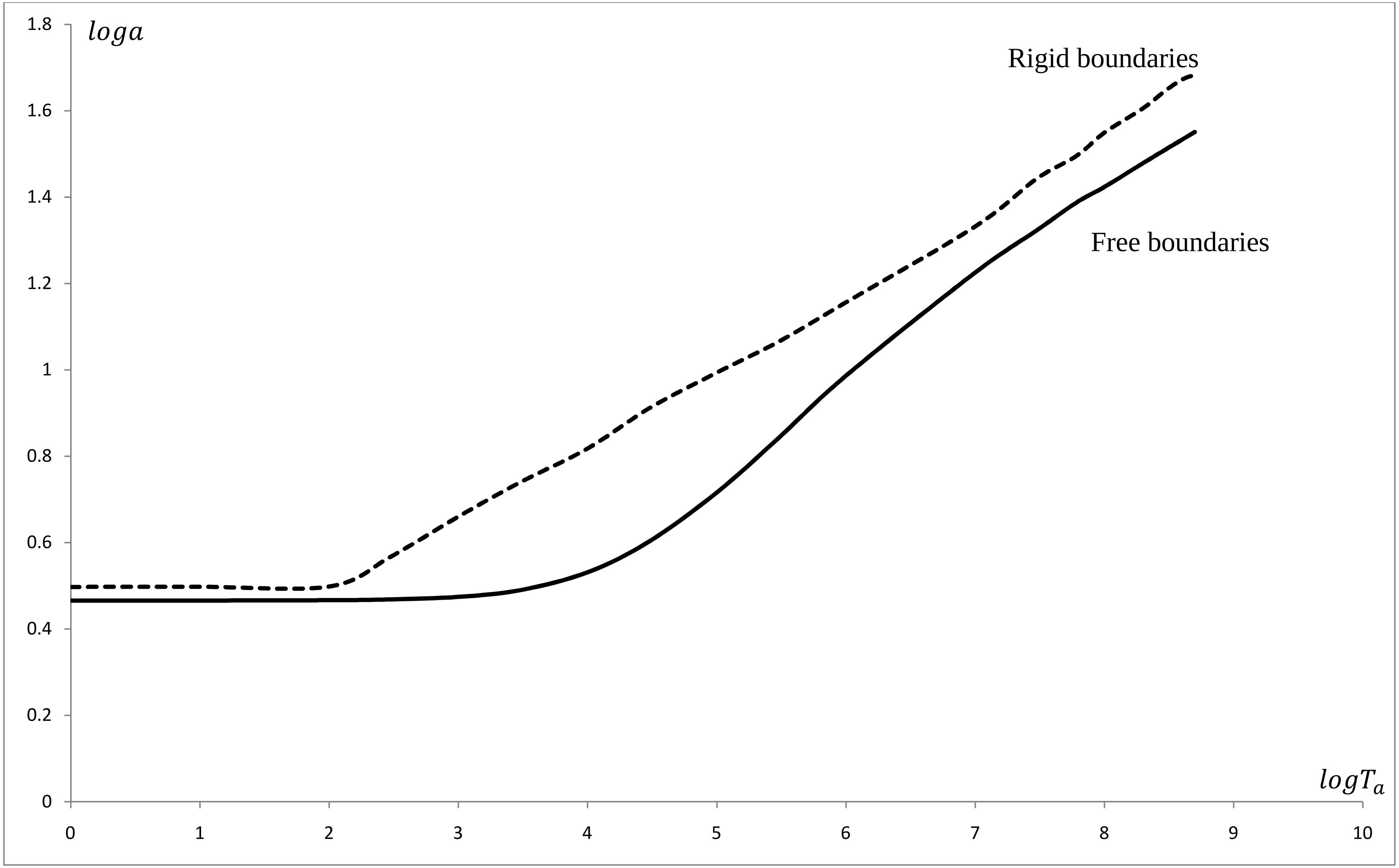

Figure 26. A comparison between free and rigid boundaries for the stationary stability case when $N = 0.01$.

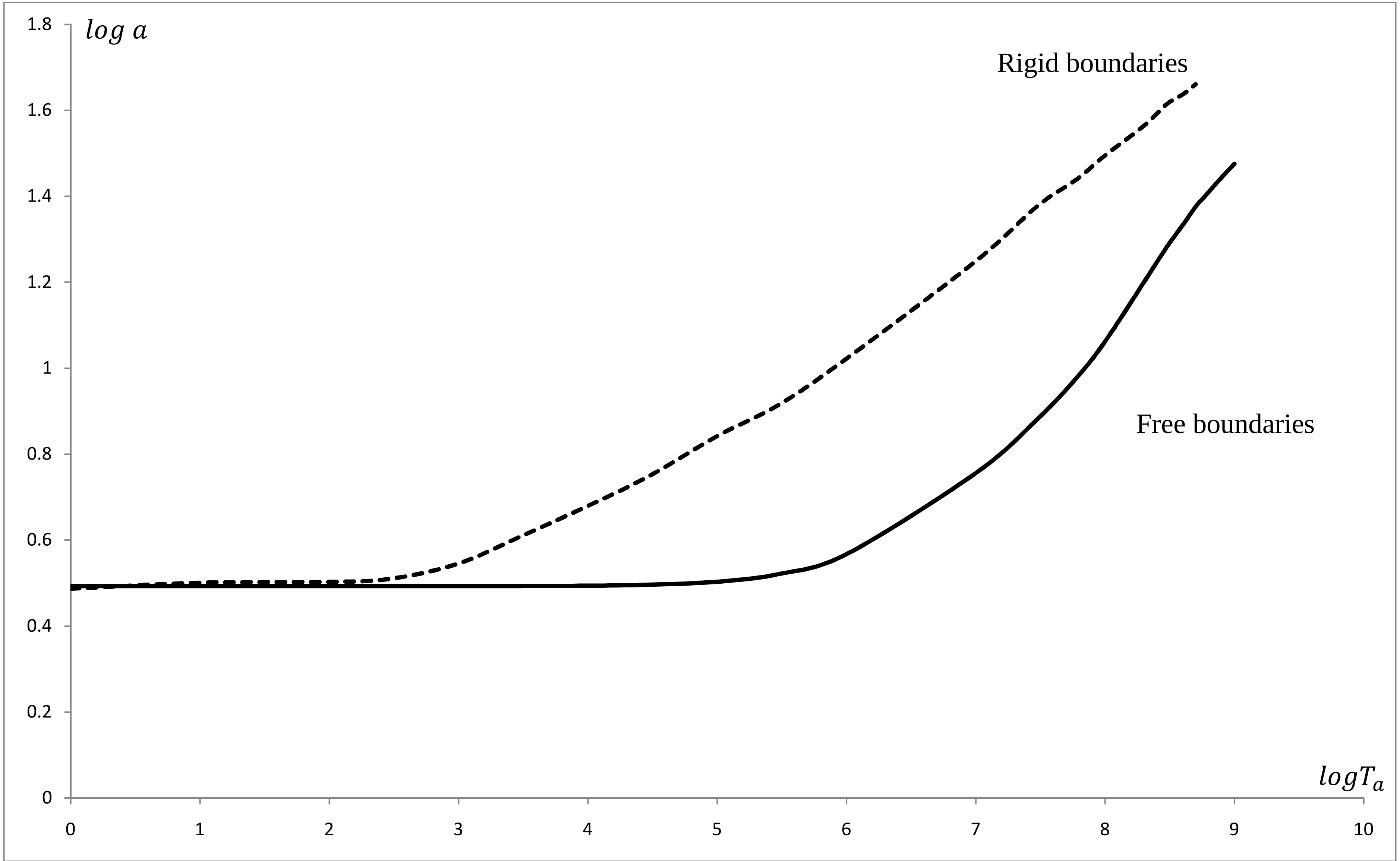

Figure 27. A comparison between free and rigid boundaries for the stationary stability case when $N = 0.001$.

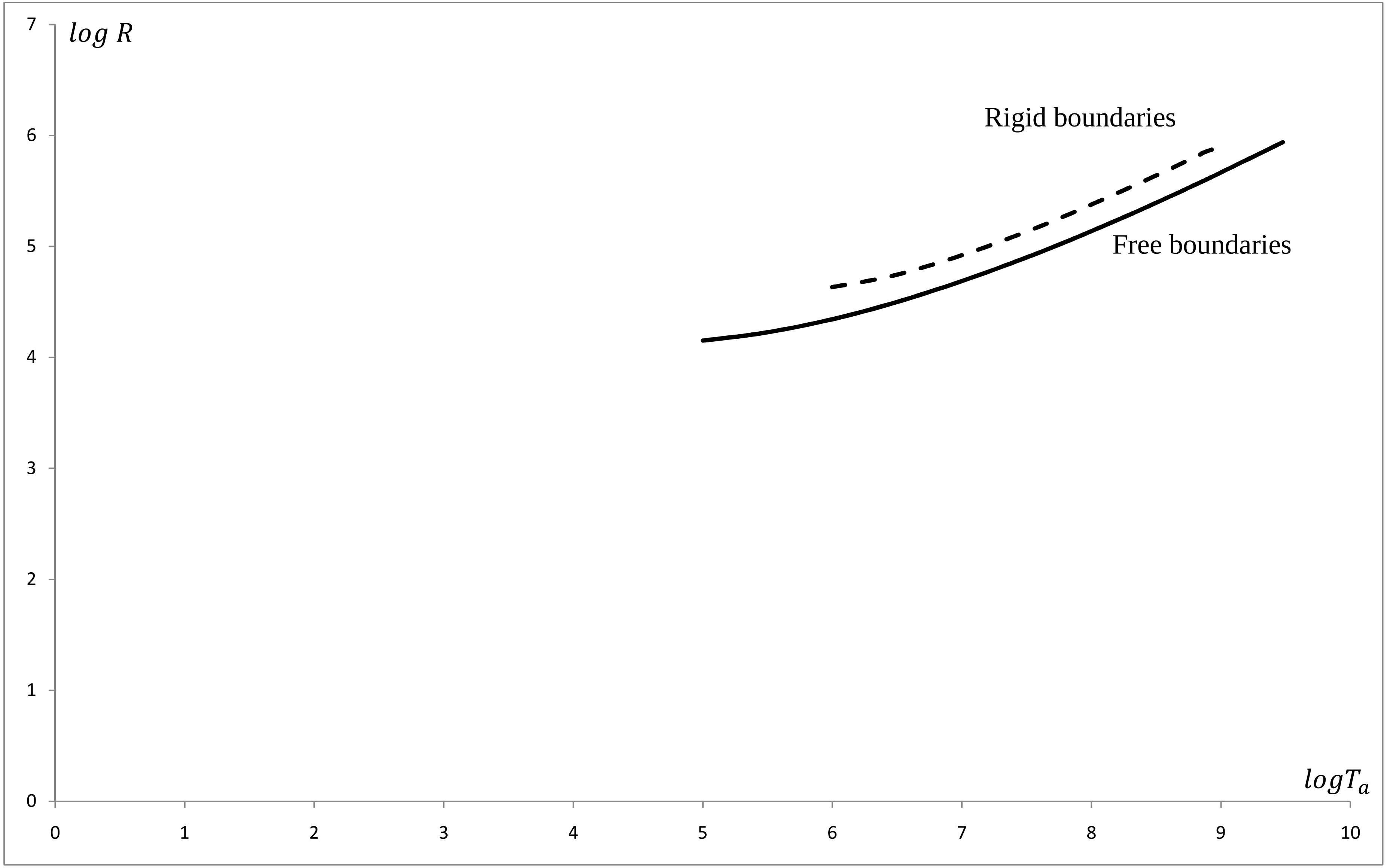

Figure 28. A comparison between the free and rigid boundaries for the overstability case for $N = 0.01$. Here $P_r = 0.05$.

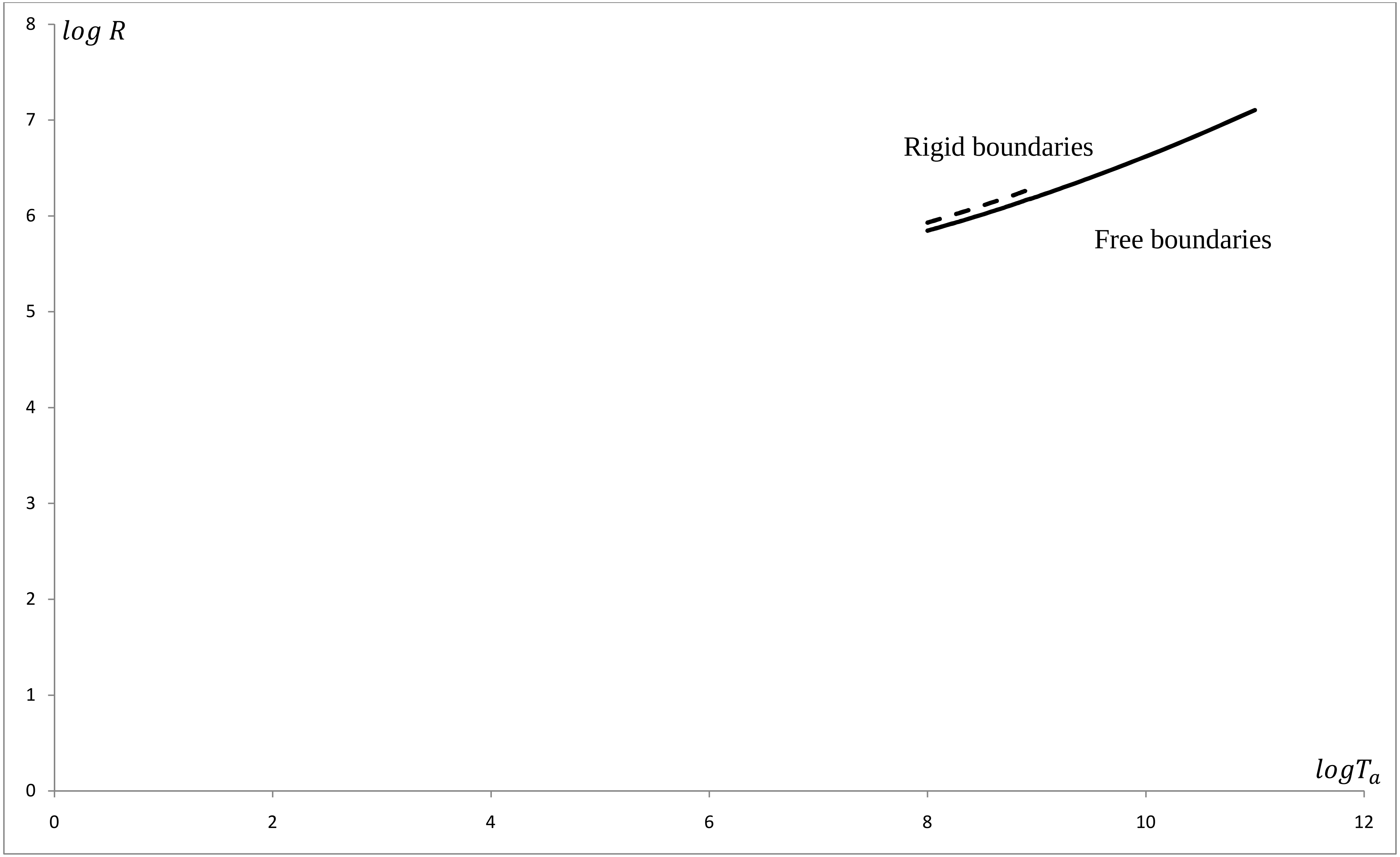

Figure 29. A comparison between the free and rigid boundaries for the overstability case for $N = 0.001$. Here $P_r = 0.05$.

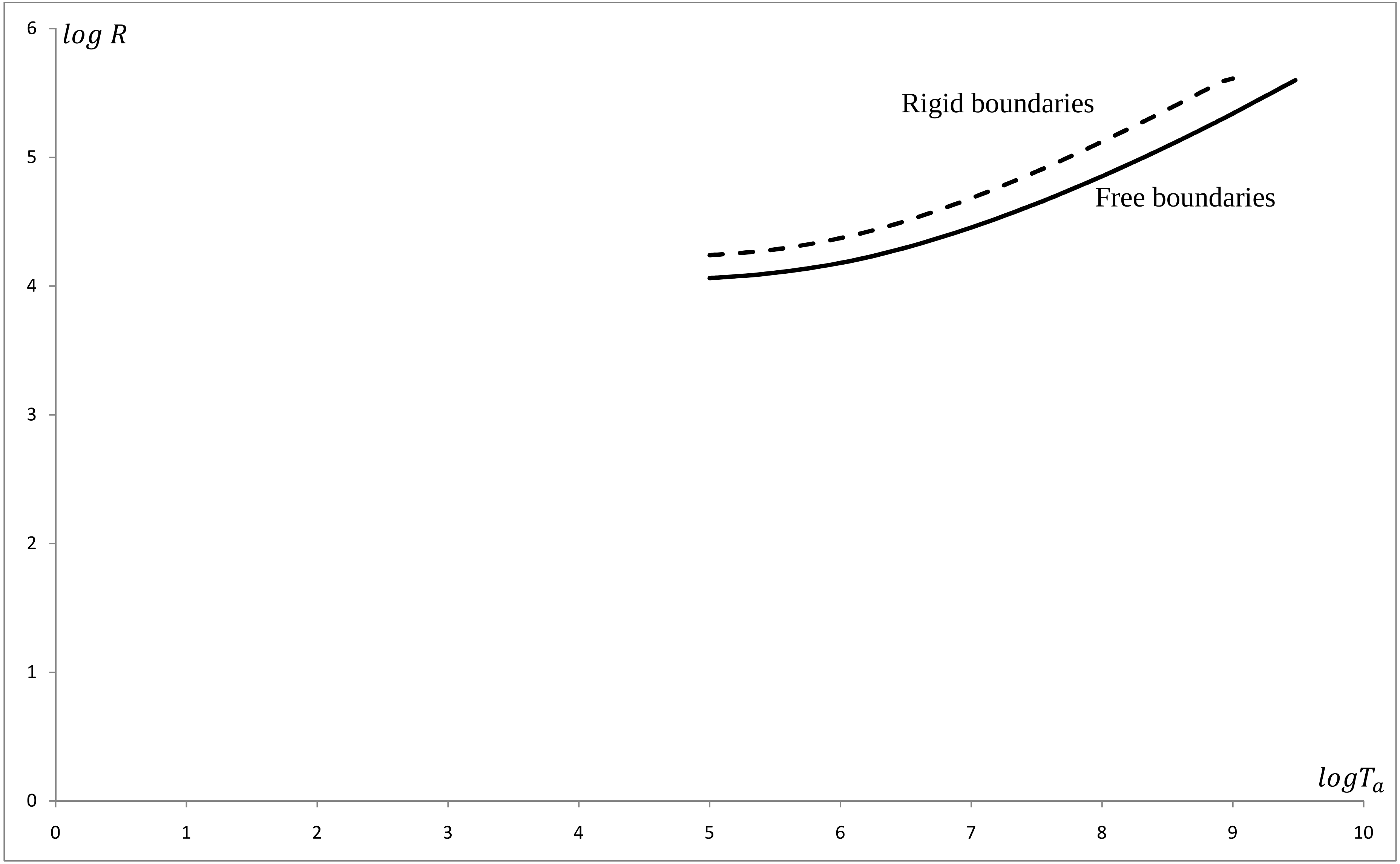

Figure 30. A comparison between the free and rigid boundaries for the overstability case when $N = 0.01$. Here $P_r = 0.025$.

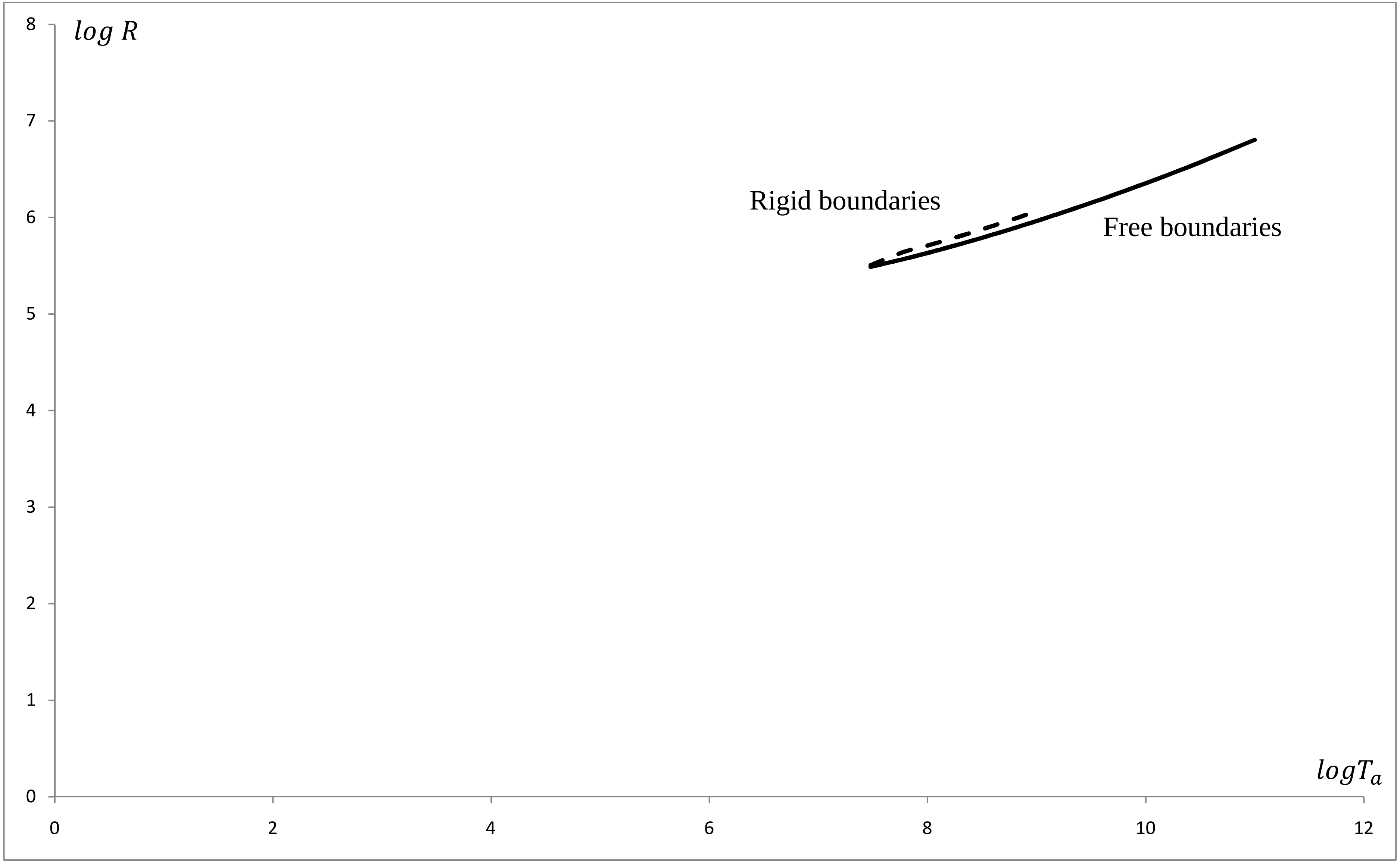

Figure 31. A comparison between the free and rigid boundaries for the overstability case when $N = 0.001$. Here $P_r = 0.025$.

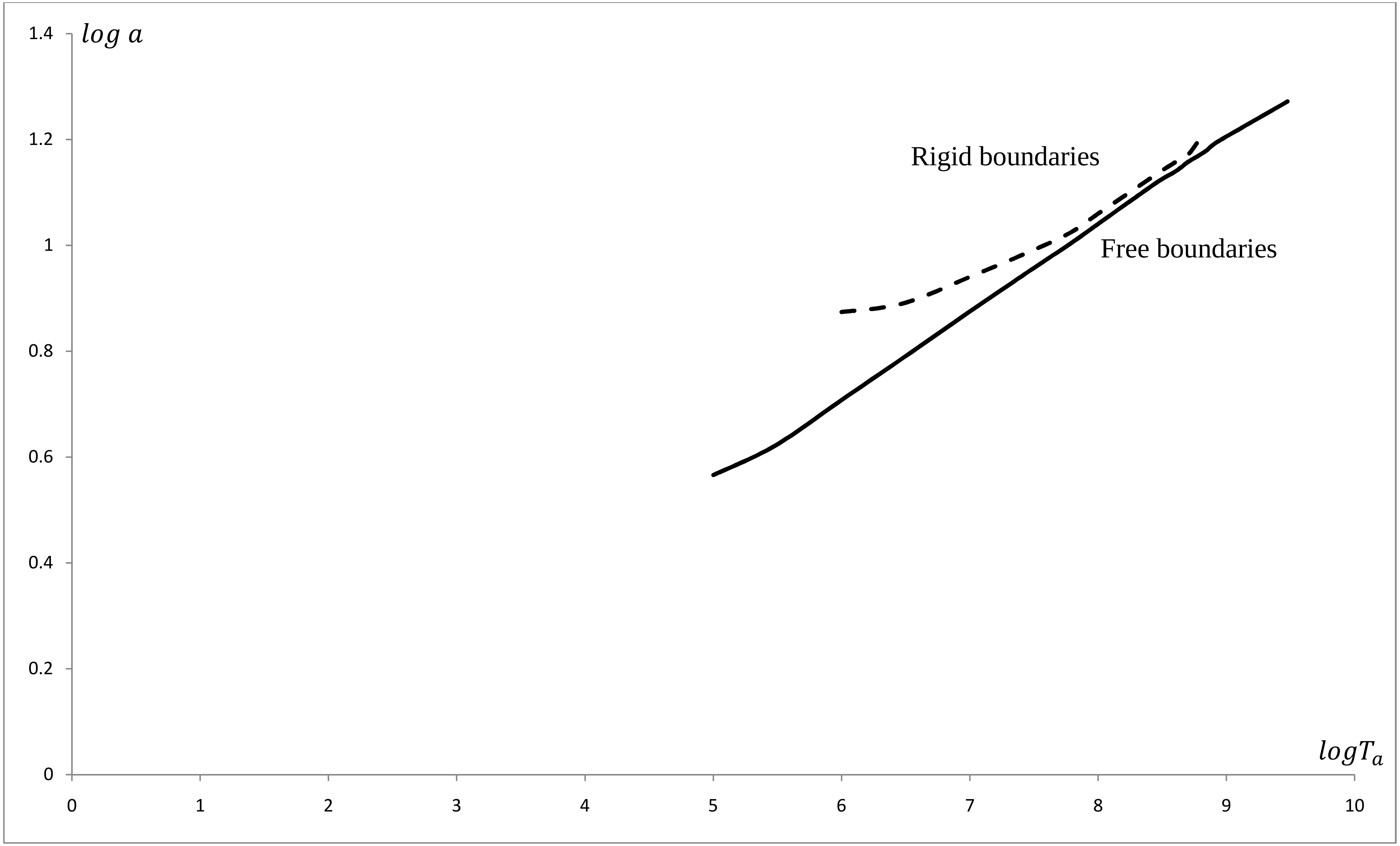

Figure 32. A comparison between the free and rigid boundaries for the overstability case for $N = 0.01$. Here $P_r = 0.05$.

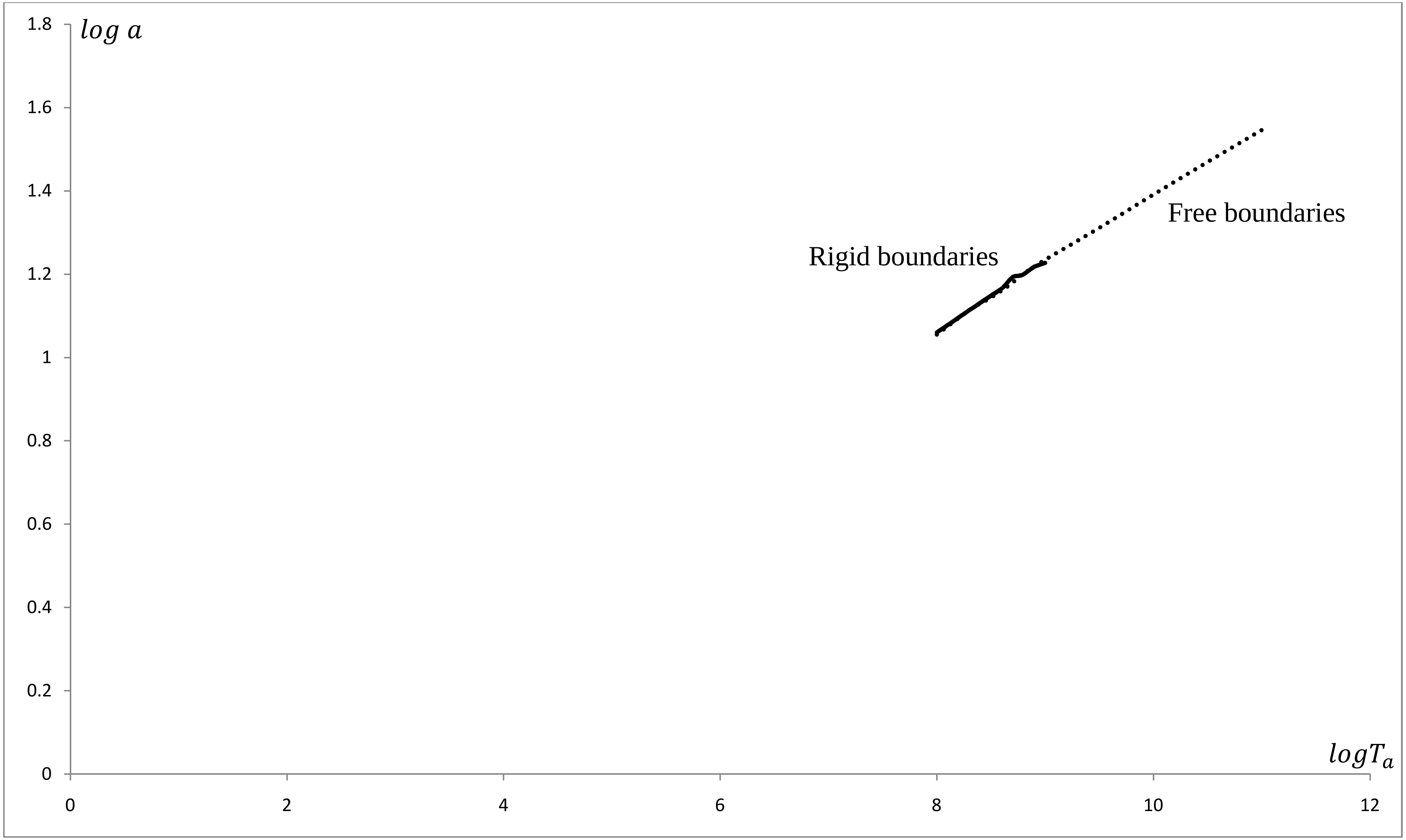

Figure 33. A comparison between the free and rigid boundaries for the overstability case for $N = 0.001$. Here $P_r = 0.05$.

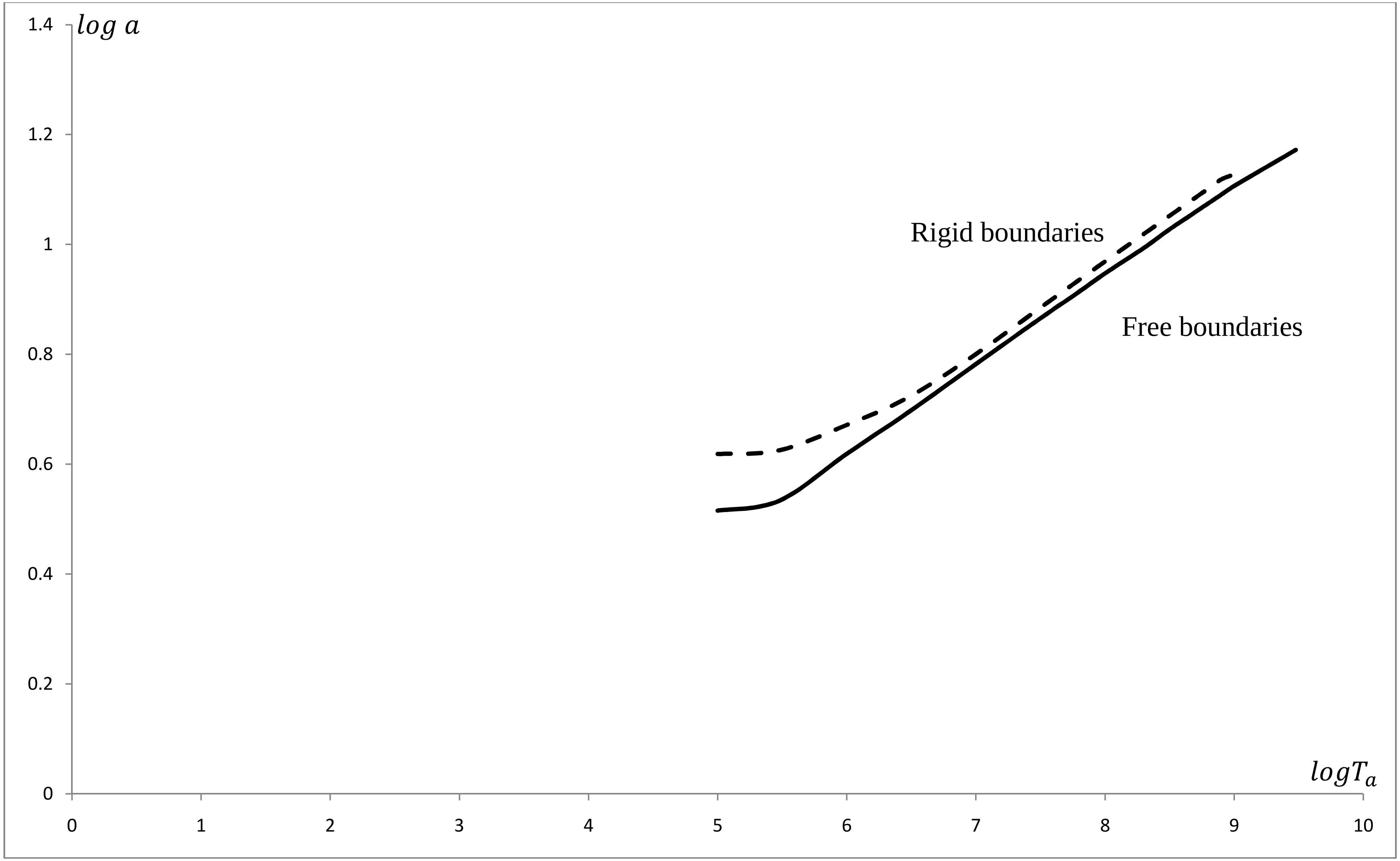

Figure 34. A comparison between the free and rigid boundaries for the overstability case for $N = 0.01$. Here $P_r = 0.025$.

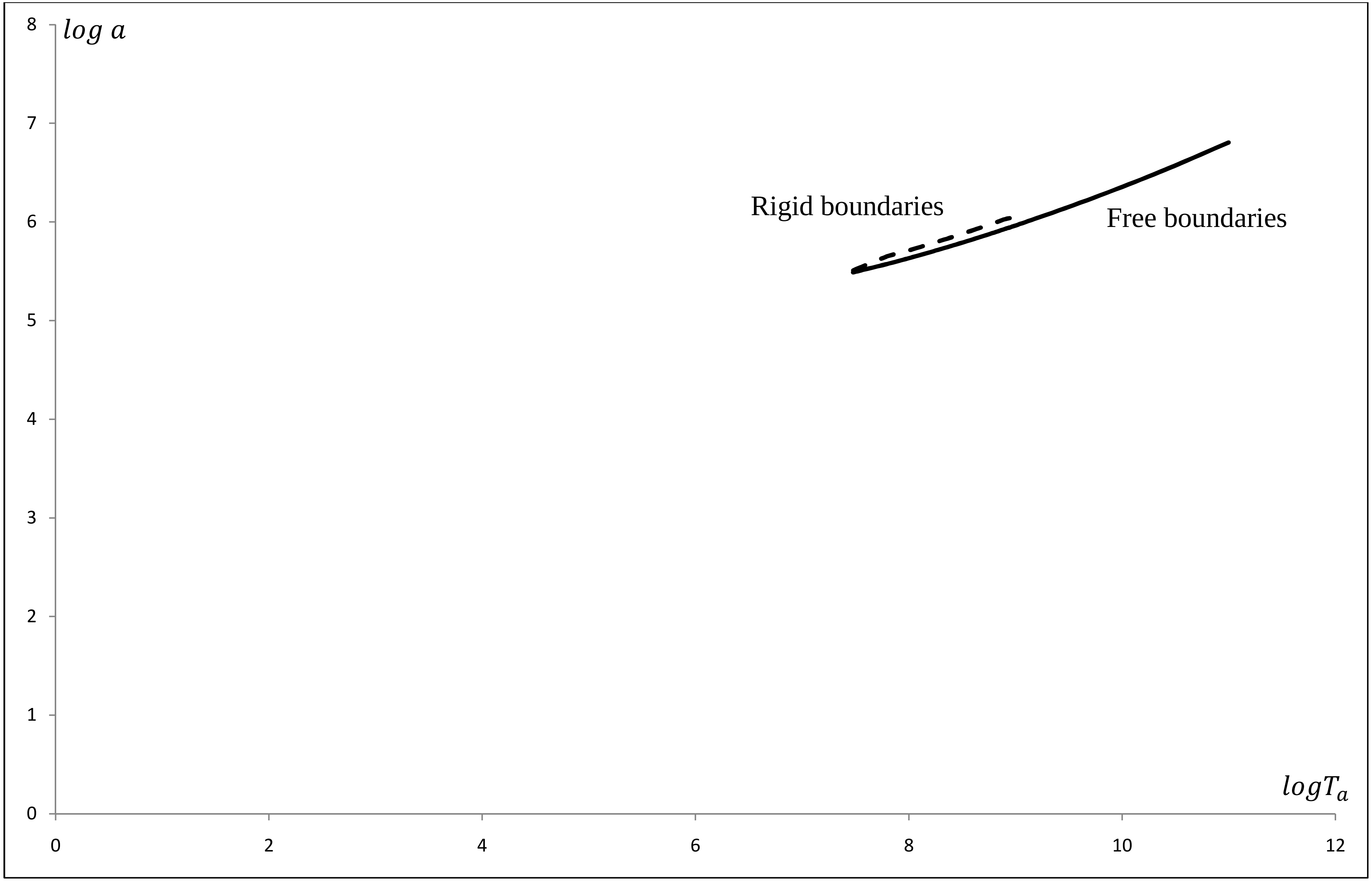

Figure 35. A comparison between the free and rigid boundaries for the overstability case for $N = 0.001$. Here $P_r = 0.025$.

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Appendix 1

APPENDIX 1

```
*PROGRAMME OF CLASSICAL BENARD PROBLEM WITH ROTATION    *
*________________________________________________________*
*THERMAL CONVECTION IN A HORIZONTAL LAYER HEATED FROM   *
*BELOW AND AFFECTED BY ROTATION ABOUT THE VERTICAL AXIS.*
*THE  BOUNDARY CONDITIONS ARE FREE (W=FI=THETA=0,DXI=0) AT*
*Z=0,Z=1.  IN  THIS PROGRAMME APPROXIMATION OF CHEBYSHEV    *
*POLYNOMIALS IS USED TO SOLVE THE PROBLEM.                 *
*________________________________________________________*
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (E =1.0D-9)
      COMMON N,R1,R2,TAY,PRV
      OPEN (8,FILE='C:\WORK\THETA\RESULTA1')

*     WHERE:(N) IS THE NUMBER OF POLYNOMIALS
*      (A & B) ARE TOW VALUES FOR THE WAVE NUMBER
*      (R1 & R2) ARE TOW VALUES FOR THE RAYLEIGH NUMBER
*      (PRV) THE VISCOUS PRANDTLE NUMBER
*      (TAY) IS THE TAYLOR NUMBER
*      (A1 & A2) AER TWO PARTICULAR POINTS

      WRITE (8,*)'INTER N,A,B,R1,R2,PRV,TAY'
      READ*, N,A,B,R1,R2,PRV,TAY
**********************************
*     RGOLD SECTION SEARCH      *
**********************************
      RGOLD =(DSQRT(5.0D0)-1.0D0)/2.0D0
      M=DABS(DLOG(E/(B-A))/(DLOG(RGOLD)))
      A1=A+RGOLD*(B-A)
      CALL CHEB (A1,U,G1R,G1I)
      A2= A+ rgold**2*(B-A)
      CALL CHEB (A2,V,G2R,G2I)
      DO 1 I=1,M
      IF (U.GT.V)THEN
      U=V
      B=A1
      A1=A2
      A2= A+ RGOLD**2*(B-A)
      CALL CHEB (A2,V,G2R,G2I)
      ELSE
      V=U
      A=A2
      A2=A1
```

```
      A1=A+RGOLD*(B-A)
      CALL CHEB (A1,U,G1R,G1I)
      END IF
      IF (DABS(U-V) .LE.E) GO TO 50
1     CONTINUE
50    AMIN=(A1 +A2)/2.0D0

      CALL CHEB (AMIN,R,SEGMAR,SEGMAI)
      WRITE (8,*) 'THE CRITICAL VALUE OF THE WAVE NUMBER =',AMIN
      WRITE (8,*) 'THE RAYEILGH NUMBER =',R
      WRITE (8,*) 'THE REAL PART OF EIGENVALUE=',SEGMAR
      WRITE (8,*) 'THE IMAGINARY PART OF EIGEVALUE=',SEGMAI
      STOP
      END
*****************************************************************
*THIS SUBROUTINE IS USED TO EVALUATE THE RAYLEIGH NUMBER  *
*****************************************************************
      SUBROUTINE CHEB (A,R,SEGMAR,SEGMAI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON N,R1,R2,TAY,PRV
      PARAMETER (L1=30,L=4*L1)
      DIMENSION F(L,L),E(L,L),XI(L1,L1),D(L1,L1),D2(L1,L1),V(L1,L1)
      DIMENSION Z(L,L),ITER(L),ALFR(L),ALFI(L),BETA(L),GR(L)
       DIMENSION  GI(L),ELANDAR(L),ELANDAI(L),ST(100)
      LOGICAL MATZ
      EXTERNAL F02BJF
      ST(1)=R1
      ST(2)=R2
      NL1=N+1
      NL2=N+2
      NL3=N+3
      N2=N*2
      N2L1=N2+1
      N2L2=N2+2
      N2L3=N2+3
      N3=N*3
      N3L1=N3+1
      N3L2=N3+2
      N3L3=N3+3
      N4=N*4
```

```
**********************************
*        THE IDENITY MATREX  (XI)   *
**********************************
         DO 3 I=1,N
         DO 2 J=1,N
         IF (I.EQ.J) THEN
         XI(I,J)= 1.0D0
         ELSE
         XI(I,J)=0.0D0
         END IF
2        CONTINUE
3        CONTINUE

         CALL DRIVE (N,D)
         CALL PRO (N,D,D,D2)
         DO 5 I=1,N
         DO 4 J=1,N
4        V(I,J)= 4.0D0*(D2(I,J)-(A/2.0D0)**2*XI(I,J))
5        CONTINUE


*****************************************************
*                  THE SECANT METHOD                 *
*****************************************************
         DO 33 K=1,100
         IF (K.GE.3) THEN
         P=(ST(K-2)*GR(K-1)- ST(K-1)*GR(K-2))/(GR(K-1)- GR(K-2))
         ST(K)=DABS(P)
         END IF

*************************************
*        THE F & E ZEROING MATRICES *
*************************************
         DO 7 I=1,N4
         DO 6 J=1,N4
         F(I,J)=0.0D0
         E(I,J)=0.0D0
6        CONTINUE
7        CONTINUE
***********************************
*    THE ELEMENTS [F11 & E11]       *
***********************************
         DO 9 I=3,N
         DO 8 J=1,N
```

```
          F(1,J)=1.0D0
          F(2,J)=(-1.0D0)**(J+1)
          F(I,J)=V(I-2,J)
          E(I,J)=XI(I-2,J)
8         CONTINUE
9         CONTINUE

************************************
*         THE ELEMENTS [F22 & E22]     *
************************************
          DO 11 I=NL3,N2
          DO 10 J=NL1,N2
          F(NL1,J)=1.0D0
          F(NL2,J)=(-1.0D0)**(J-NL1)
          F(I,J)=V(I-NL2,J-N)
          E(I,J)=PRV*XI(I-NL2,J-N)
10        CONTINUE
11        CONTINUE
********************************
*         THE ELEMENTS [ F33]       *
********************************
          DO 12 I=1,N
          F(N2L1,I+N2)=2.0D0* DBLE((I-1)**2)
          F(N2L2,I+N2)=2.0D0* DBLE((I-1)**2)*(-1.0D0)**(I)
12        CONTINUE
          DO 14 I=N2L3,N3
          DO 13 J=N2L1,N3
          F(I,J)= V(I-N2L2,J-N2)
13        CONTINUE
14        CONTINUE
********************************
*         THE ELEMENTS [ E33]       *
********************************
          DO 16 I=N2L3,N3
          DO 15 J=N2L1,N3
          E(I,J)=XI(I-N2L2,J-N2)
15        CONTINUE
16        CONTINUE

********************************
*         THE ELEMENTS [ F44]       *
********************************
          do 18 I=N3L3,N4
          DO 17 J=N3L1,N4
```

```
      F(N3L1,J)=1.0D0
      F(N3L2,J)=(-1.0D0)**(J-NL1)
      F(I,J)=V(I-N3L2,J-N3)
17    CONTINUE
18    CONTINUE
*******************************
*     THE ELEMENTS [ F12]     *
*******************************
      DO 20 I=3,N
      DO 19 J=NL1,N2
      F(I,J)=(-1.0D0)*DSQRT(ST(K))*(A**2)*XI(I-2,J-N)
19    CONTINUE
20    CONTINUE
*******************************
*     THE ELEMENTS [ F13]     *
*******************************
      DO 45 I=3,N
      DO 44 J=N2L1,N3
      F(I,J)=(-2.0D0)*DSQRT(TAY)*D(I-2,J-N2)
44    CONTINUE
45    CONTINUE
*******************************
*     THE ELEMENTS [ F24]     *
*******************************
      DO 22 I=NL3,N2
      DO 21 J=N3L1,N4
      F(I,J)=DSQRT (ST(K))*XI(I-NL2,J-N3)
21    CONTINUE
22    CONTINUE
******************************
*     THE ELEMENTS [F34]     *
******************************
      DO 24 I=N2L3,N3
      DO 23 J=N3L1,N4
      F(I,J)=2.0D0*DSQRT(TAY)*D(I-N2L2,J-N3)
23    CONTINUE
24    CONTINUE
*******************************
*     THE ELEMENTS [ F41]     *
*******************************
      DO 26 I=N3L3,N4
      DO 25 J=1,N
      F(I,J)=(-1.0D0)*XI(I-N3L2,J)
25    CONTINUE
```

```
26      CONTINUE

*******************************************
*       WE WILL USE NAGROUTINE (F02BJF)     *
*******************************************
        MATZ=.FALSE.
        EPS1 =(10.0d0)**(-15)
        IFAIL=0
        CALL F02BJF (N4,F,L,E,L,EPS1,ALFR,ALFI,BETA,MATZ,Z,L,ITER,IFAIL)
        J=0
        DO 27 I=1,N4
        IF (BETA(I).NE.0.0D0) THEN
        J=1+J
        ELANDAR(J)=ALFR(I)/BETA(I)
        ELANDAI(J)=ALFI(I)/BETA(I)
        END IF
27      CONTINUE
*       CHOOSING THE LARGEST REAL PART OF THE EIGENVALUE

        ELARGER=ELANDAR(1)
        DO 28 I=2,J
        IF (ELANDAR(I).GT.ELARGER) THEN
        ELARGER=ELANDAR(I)
        ELARGEI=ELANDAI(I)
        END IF
28      CONTINUE
        GR(K)=ELARGER
        GI(K)=ELARGEI
        IF (K.LT.2) GO TO 33
        IF (DABS(ST(K)-ST(K-1)).LT.1.0D-7)THEN
        R=ST(K)
        SEGMAR=GR(K)
        SEGMAI=GI(K)
        RETURN
        END IF
33      CONTINUE
        END
******************************************************************
* THIS SUBROUTINE CALCULATES THE PRODUCT OF TWO MATRICES  *
******************************************************************
        SUBROUTINE PRO(N,H,T,Z)
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
        PARAMETER (L1=30)
```

```
      DIMENSION H(L1,L1),T(L1,L1),Z(L1,L1)
      DO 32,I=1,N
      DO 31 ,J=1,N
      Z(I,J)=0.0D0
      DO 30 K=1,N
      Z(I,J)=Z(I,J)+H(I,K)*T(K,J)
30    CONTINUE
31    CONTINUE
32    CONTINUE
      RETURN
      END
*****************************************************************
*     THIS SUBROUTINE BULDS THE DERIVATIVE MATRIX                *
*****************************************************************
      SUBROUTINE DRIVE (N,D)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (L1=30)
      DIMENSION D(L1,L1)
      DO 35 I=1,N
      DO 35 J=1,N
      IF ((J.GT.I).AND.(MOD(J-I,2).EQ.1))THEN
      D(I,J)=2.0D0* DBLE(J-1)
      ELSE
      D(I,J)=0.0D0
      END IF
35    CONTINUE
      DO 34 I=1,N
      D(1,I)=0.5D0*D(1,I)
34    CONTINUE
      RETURN
      END
```

Appendix 2

APPENDIX 2

```
* PROGRAMME OF CLASSICAL BENARD PROBLEM WITH ROTATION          *
*________________________________________________________________*
* THERMAL CONVECTION IN A HORIZONTAL LAYER HEATED FROM          *
*BELOW AND AFFECTED BY * *ROTATION ABOUT THE VERTICAL AXIS.     *
*THE BOUNDARY CONDITIONS ARE RIGID (W=THETA=0,XI=DW=0) AT       *
*Z=0, Z=1. IN THIS PROGRAMME APPROXIMATION OF CHEBYSHEV         *
*POLYNOMIALS IS USEDTO SOLVE THEPROBLEM.                        *
*________________________________________________________________*
      IMPLICIT DOUBLE PRECISION  (A-H,O-Z)
      PARAMETER (E=1.0D-9)
      COMMON N,R1,R2,TAY,PRV
      OPEN (8,FILE='C:\WORK\THETA\RESULTA1')

*     WHERE:(N) IS THE NUMBER OF POLYNOMIALS
*      (A & B) ARE TOW VALUES FOR THE WAVE NUMBER
*      (R1 & R2) ARE TOW VALUES FOR THE RAYLEIGH NUMBER
*      (PRV) THE VISCOUS PRANDTLE NUMBER
*      (TAY) IS THE TAYLOR NUMBER
*      (A1 & A2) AER TWO PARTICULAR POINTS

      WRITE (8,*)'INTER N,A,B,R1,R2,PRV,TAY'
      READ*, N,A,B,R1,R2,PRV,TAY
************************************
*     RGOLD SECTION SEARCH       *
************************************
      RGOLD =(DSQRT(5.0D0)-1.0D0)/2.0D0
      M=DABS(DLOG(E/(B-A))/(DLOG(RGOLD)))
      A1=A+RGOLD*(B-A)
      CALL CHEB (A1,U,G1R,G1I)

      A2= A+RGOLD **2*(B-A)
      CALL CHEB (A2,V,G2R,G2I)

      DO 1 I=1,M
      IF (U.GT.V)THEN
      U=V
      B=A1
      A1=A2
```

```
      A2= A+ RGOLD**2*(B-A)

      CALL CHEB (A2,V,G2R,G2I)
      ELSE
      V=U
      A=A2
      A2=A1
      A1=A+RGOLD*(B-A)

      CALL CHEB (A1,U,G1R,G1I)
      END IF
      IF (DABS(U-V) .LE.E) GO TO 50
1     CONTINUE
50    AMIN=(A1 +A2)/2.0D0

      CALL CHEB (AMIN,R,SEGMAR,SEGMAI)
      WRITE (8,*) 'THE CRITICAL VALUE OF THE WAVE NUMBER =',AMIN
      WRITE (8,*) 'THE RAYEILGH NUMBER =',R
      WRITE (8,*) 'THE REAL PART OF EIGENVALUE=',SEGMAR
      WRITE (8,*) 'THE IMAGINARY PART OF EIGEVALUE=',SEGMAI
      STOP
      END
**********************************************************************
*THIS SUBROUTINE IS USED TO EVALUATE THE RAYLEIGH NUMBER            *
**********************************************************************
      SUBROUTINE CHEB (A,R,SEGMAR,SEGMAI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON N,R1,R2,TAY,PRV
      PARAMETER (L1=30,L=4*L1)
      DIMENSION F(L,L),E(L,L),XI(L1,L1),D(L1,L1),D2(L1,L1),V(L1,L1)
      DIMENSION Z(L,L),ITER(L),ALFR(L),ALFI(L),BETA(L),GR(L)
      DIMENSION  GI(L),ELANDAR(L),ELANDAI(L),ST(100)
      LOGICAL MATZ

      EXTERNAL F02BJF
      ST(1)=R1
      ST(2)=R2
      NL1=N+1
      NL2=N+2
      NL3=N+3
      N2=N*2
      N2L1=N2+1
      N2L2=N2+2
      N2L3=N2+3
```

```
      N3=N*3
      N3L1=N3+1
      N3L2=N3+2
      N3L3=N3+3
      N4=N*4
*************************************
*     THE IDENITY MATREX  (XI)      *
*************************************
      DO 3 I=1,N
      DO 2 J=1,N
      IF (I.EQ.J) THEN
      XI(I,J)= 1.0D0
      ELSE
      XI(I,J)=0.0D0
      END IF
2     CONTINUE
3     CONTINUE

      CALL DRIVE (N,D)
      CALL PRO (N,D,D,D2)

      DO 5 I=1,N
      DO 4 J=1,N
4     V(I,J)= 4.0D0*(D2(I,J)-(A/2.0D0)**2*XI(I,J))
5     CONTINUE

*************************************************
*             THE SECANT METHOD                 *
*************************************************
      DO 33 K=1,100
      IF (K.GE.3) THEN
      P=(ST(K-2)*GR(K-1)- ST(K-1)*GR(K-2))/(GR(K-1)- GR(K-2))
      ST(K)=DABS(P)
      END IF

**************************************
*     THE F & E ZEROING MATRICES     *
**************************************
      DO 7 I=1,N4
      DO 6 J=1,N4
      F(I,J)=0.0D0
      E(I,J)=0.0D0
6     CONTINUE
7     CONTINUE
```

```
***********************************
*   THE ELEMENTS [F11 & E11]        *
***********************************
      DO 9 I=3,N
      DO 8 J=1,N
      F(I,J)=V(I-2,J)
      E(I,J)=XI(I-2,J)
8     CONTINUE
9     CONTINUE
**********************************
*     THE ELEMENTS [F22 & E22]    *
**********************************
      DO 11 I=NL3,N2
      DO 10 J=NL1,N2
      F(NL1,J)=1.0D0
      F(NL2,J)=(-1.0D0)**(J-NL1)
      F(I,J)=V(I-NL2,J-N)
      E(I,J)=PRV*XI(I-NL2,J-N)
10    CONTINUE
11    CONTINUE
*********************************
*     THE ELEMENTS [ F33]        *
*********************************
      DO 14 I=N2L3,N3
      DO 13 J=N2L1,N3
      F(N2L1,J)=1.0D0
      F(N2L2,J)=(-1.0D0)**(J-N2L1)
      F(I,J)= V(I-N2L2,J-N2)
13    CONTINUE
14    CONTINUE
*********************************
*     THE ELEMENTS [ E33]        *
*********************************
      DO 16 I=N2L3,N3
      DO 15 J=N2L1,N3
      E(I,J)=XI(I-N2L2,J-N2)
15    CONTINUE
16    CONTINUE
********************************
*     THE ELEMENTS [ F44]       *
********************************
      DO 18 I=N3L3,N4
      DO 17 J=N3L1,N4
      F(N3L1,J)=1.0D0
```

```
      F(N3L2,J)=(-1.0D0)**(J-N3L1)
      F(I,J)=V(I-N3L2,J-N3)
17    CONTINUE
18    CONTINUE
*********************************
*     THE ELEMENTS [ F12]       *
*********************************
      DO 20 I=3,N
      DO 19 J=NL1,N2
      F(I,J)=(-1.0D0)*DSQRT(ST(K))*(A**2)*XI(I-2,J-N)
19    CONTINUE
20    CONTINUE
*********************************
*     THE ELEMENTS [ F13]       *
*********************************
      DO 45 I=3,N
      DO 44 J=N2L1,N3
      F(I,J)=(-2.0D0)*DSQRT(TAY)*D(I-2,J-N2)
44    CONTINUE
45    CONTINUE
*********************************
*     THE ELEMENTS [ F24]       *
*********************************
      DO 22 I=NL3,N2
      DO 21 J=N3L1,N4
      F(I,J)=DSQRT (ST(K))*XI(I-NL2,J-N3)
21    CONTINUE
22    CONTINUE
********************************
*     THE ELEMENTS [F34]       *
********************************
      DO 24 I=N2L3,N3
      DO 23 J=N3L1,N4
      F(I,J)=2.0D0*DSQRT(TAY)*D(I-N2L2,J-N3)
23    CONTINUE
24    CONTINUE
*********************************
*     THE ELEMENTS [ F41]       *
*********************************
      DO 26 I=N3L3,N4
      DO 25 J=1,N
      F(I,J)=(-1.0D0)*XI(I-N3L2,J)
25    CONTINUE
26    CONTINUE
```

```
*********************************
*       THE ELEMENTS [F14]              *
*********************************
        DO 60 J=1,N
        F(1,J+N3)=2.0D0* DBLE((J-1)**2)
        F(2,J+N3)=2.0D0* DBLE((J-1)**2)*(-1.0D0)**(J)
60      CONTINUE
********************************************
*       WE WILL USE NAGROUTINE (F02BJF)     *
********************************************
        MATZ=.FALSE.
        EPS1 =(10.0D0)**(-15)
        IFAIL=0

        CALL F02BJF (N4,F,L,E,L,EPS1,ALFR,ALFI,BETA,MATZ,Z,L,ITER,IFAIL)
        J=0
        DO 27 I=1,N4
        IF (BETA(I).NE.0.0D0) THEN
        J=1+J
        ELANDAR(J)=ALFR(I)/BETA(I)
        ELANDAI(J)=ALFI(I)/BETA(I)
        END IF
27      CONTINUE
*       CHOOSING THE LARGEST REAL PART OF THE EIGENVALUE
        ELARGER=ELANDAR(1)
        DO 28 I=2,J
        IF (ELANDAR(I).GT.ELARGER) THEN
        ELARGER=ELANDAR(I)
        ELARGEI=ELANDAI(I)
        END IF
28      CONTINUE
        GR(K)=ELARGER
        GI(K)=ELARGEI
        IF (K.LT.2) GO TO 33
        IF (DABS(ST(K)-ST(K-1)).LT.1.0D-7)THEN
        R=ST(K)
        SEGMAR=GR(K)
        SEGMAI=GI(K)
        RETURN
        END IF
33      CONTINUE
        END
```

```
*****************************************************************
*THIS SUBROUTINE CALCULATES THE PRODUCT OF TWO MATRICES  *
*****************************************************************
      SUBROUTINE PRO(N,H,T,Z)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (L1=50)
      DIMENSION H(L1,L1),T(L1,L1),Z(L1,L1)
      DO 32,I=1,N
      DO 31 ,J=1,N
      Z(I,J)=0.0D0
      DO 30 K=1,N
      Z(I,J)=Z(I,J)+H(I,K)*T(K,J)
30    CONTINUE
31    CONTINUE
32    CONTINUE
      RETURN
      END
************************************************************
*     THIS SUBROUTINE BULDS THE DERIVATIVE  MATRIX       *
************************************************************
      SUBROUTINE DRIVE (N,D)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (L1=50)
      DIMENSION D(L1,L1)
      DO 35 I=1,N
      DO 35 J=1,N
      IF ((J.GT.I).AND.(MOD(J-I,2).EQ.1))THEN
      D(I,J)=2.0D0* DBLE(J-1)
      ELSE
      D(I,J)=0.0D0
      END IF
35    CONTINUE
      DO 34 I=1,N
      D(1,I)=0.5D0*D(1,I)
34    CONTINUE
      RETURN
      END
```

Appendix 3

APPENDIX 3

```
************************************************************************
*PROGRAMME OF  BENARD PROBLEM WITH ROTATION AND FREE                   *
*BOUNDARY                                                              *
*______________________________________________________________________*
*THERMAL CONVECTION IN A HORIZONTAL POROUS LAYER HEATED                *
*FROM BELOW  AND AFFECTED BY ROTATION ABOUT THE VERTICAL               *
*AXIS. THE BOUNDARY CONDITIONS ARE FREE (W=FI=THETA=0,DXI=0)           *
*AT Z=0,Z=1. IN THIS PROGRAMME APPROXIMATION OF CHEBYSHEV              *
*POLYNOMIALS IS USED TO SOLVE THE PROBLEM.                             *
*______________________________________________________________________*
      IMPLICIT DOUBLE PRECISION  (A-H,O-Z)
      PARAMETER (E=1.0D-9)
      COMMON N,R1,R2,PRV,PON,TAY
      OPEN (8,FILE='C:\WORK\THETA\RESULTA2FREE')

*     WHERE: (N) IS THE NUMBER OF POLYNOMIALS
*     (A & B) ARE TOW VALUES FOR THE WAVE NUMBER
*     (R1 & R2) ARE TOW VALUES FOR THE RAYLEIGH NUMBER
*     (PRV) THE VISCOUS PRANDTLE NUMBER
*     (PON) IS THE INVERSE OF THE DIMENSIONLESS PERMEABILITY
*     (TAY) IS THE TAYLOR NUMBER
*     (A1 & A2) AER TWO PARTICULAR POINTS

      WRITE(8,*)'INTER N,A,B,PRV,PON,TAY
      READ*,N,A,B,R1,R2,PRV,PON,TAY
**********************************
*     RGOLD SECTION SEARCH      *
**********************************
      RGOLD =(DSQRT(5.0D0)-1.0D0)/2.0D0
      M=DABS(DLOG(E/(B-A))/(DLOG(RGOLD)))
      A1=A+RGOLD*(B-A)
      CALL CHEB (A1,U,G1R,G1I)
      A2= A+ RGOLD**2*(B-A)
      CALL CHEB (A2,V,G2R,G2I)
      DO 1 I=1,M
      IF (U.GT.V)THEN
      U=V
      B=A1
      A1=A2
      A2= A+ RGOLD**2*(B-A)
      CALL CHEB (A2,V,G2R,G2I)
```

```
      ELSE
      V=U
      A=A2
      A2=A1
      A1=A+RGOLD*(B-A)
      CALL CHEB (A1,U,G1R,G1I)
      END IF

      IF (DABS(U-V) .LE.E) GO TO 50
1     CONTINUE
50     AMIN=(A1 +A2)/2.0D0
      CALL CHEB (AMIN,R,SEGMAR,SEGMAI)
      WRITE(8,*) 'THE CRITICAL VALUE OF THE WAVE NUMBER =',AMIN
      WRITE(8,*) 'THE THERMAL RAYEILGH NUMBER =',R
      WRITE(8,*) 'THE REAL PART OF EIGENVALUE=',SEGMAR
      WRITE(8,*) 'THE IMAGINARY PART OF EIGEVALUE=',SEGMAI
      STOP
      END
*********************************************************************
*THIS SUBROUTINE IS USED TO EVALUATE THE RAYLEIGH NUMBER *
*********************************************************************
      SUBROUTINE CHEB (A1,R,SEGMAR,SEGMAI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON N,R1,R2,PRV,PON,TAY
      PARAMETER  (L1=30,L=4*L1)
      DIMENSION  F(L,L), E(L,L), XI(L1,L1), D(L1,L1), D2(L1,L1) , V(L1,L1)
       DIMENSION  VPN(L1,L1), Z(L,L), ITER(L), ALFR(L), ALFI(L), ST(100)
       DIMENSION  BETA(L), GR(L), GI(L), ELANDAR(L), ELANDAI(L)
      LOGICAL  MATZ
      EXTERNAL F02BJF
      ST(1)= R1
      ST(2)= R2
      NL1=N+1
      NL2=N+2
      NL3=N+3
      N2=N*2
      N2L1=N2+1
      N2L2=N2+2
      N2L3=N2+3
      N3=N*3
      N3L1=N3+1
      N3L2=N3+2
      N3L3=N3+3
      N4=N*4
```

```
************************************
*       THE IDENITY MATREX  (XI)         *
************************************
        DO 3 I=1,N
        DO 2 J=1,N
        IF (I.EQ.J) THEN
         XI(I,J) = 1.0D0
        ELSE
         XI(I,J) =0.0D0
        END IF
2       CONTINUE
3       CONTINUE

        CALL DRIVE (N,D)
        CALL PRO (N,D,D,D2)

        DO 5 I=1,N
        DO 4 J=1,N
4       V(I,J)= 4.0D0*(D2(I,J)-(A1/2.0D0)**2*XI(I,J))
5       CONTINUE

        DO 7 I=1,N
        DO 6 J=1,N
6       VPN(I,J)= V(I,J)-PON*XI(I,J)
7       CONTINUE
************************************
*       THE SECANT METHOD                *
************************************
         DO 33 K=1,100
        IF (K.GE.3) THEN
        P=(ST(K-2)*GR(K-1)- ST(K-1)*GR(K-2))/(GR(K-1)- GR(K-2))
        ST(K)=DABS(P)
        END IF
*************************************
*       THE F & E ZEROING MATRICES        *
*************************************
        DO 9 I=1,N4
        DO 8 J=1,N4
        F(I,J)=0.0D0
        E(I,J)=0.0D0
8       CONTINUE
9       CONTINUE
```

```
***********************************
*   THE ELEMENTS [F11 & E11]      *
***********************************
        DO 11 I=3,N
        DO 10 J=1,N
        F(1,J)=1.0D0
        F(2,J)=(-1.0D0)**(J+1)
        F(I,J)=VPN(I-2,J)
        E(I,J)=XI(I-2,J)
10      CONTINUE
11      CONTINUE
*************************************
*       THE ELEMENTS [F22 & E22]     *
*************************************
        DO 13 I=NL3,N2
        DO 12 J=NL1,N2
        F(NL1,J)=1.0D0
        F(NL2,J)=(-1.0D0)**(J-NL1)
        F(I,J)=V(I-NL2,J-N)
        E(I,J)=PRV*XI(I-NL2,J-N)
12      CONTINUE
13      CONTINUE
**********************************
*       THE ELEMENTS [ E33]      *
**********************************
        DO 15 I=N2L3,N3
        DO 14 J=N2L1,N3
        E(I,J)=XI(I-N2L2,J-N2)
14      CONTINUE
15      CONTINUE
*********************************
*       THE ELEMENTS [ F33]     *
*********************************
        DO 16 I=1,N
        F(N2L1,I+N2)=2.0D0* DBLE((I-1)**2)
        F(N2L2,I+N2)=2.0D0* DBLE((I-1)**2)*(-1.0D0)**(I)
16      CONTINUE
        DO 18 I=N2L3,N3
        DO 17 J=N2L1,N3
        F(I,J)= VPN(I-N2L2,J-N2)
17      CONTINUE
18      CONTINUE
```

```
*********************************
*        THE ELEMENTS [ F44]        *
*********************************
         DO 20 I=N3L3,N4
         DO 19 J=N3L1,N4
         F(N3L1,J)=1.0D0
         F(N3L2,J)=(-1.0D0)**(J-NL1)
         F(I,J)=V(I-N3L2,J-N3)
19       CONTINUE
20       CONTINUE
**********************************
*        THE ELEMENTS [ F12]         *
**********************************
         DO 22 I=3,N
         DO 21 J=NL1,N2
         F(I,J)=(-1.0D0)*DSQRT(ST(K))*(A1**2)*XI(I-2,J-N)
21       CONTINUE
22       CONTINUE
**********************************
*        THE ELEMENTS [ F13]         *
**********************************
         DO 24 I=3,N
         DO 23 J=N2L1,N3
         F(I,J)=(-2.0D0)*DSQRT(TAY)*D(I-2,J-N2)
23       CONTINUE
24       CONTINUE
**********************************
*        THE ELEMENTS [ F24]         *
**********************************
         DO 26 I=NL3,N2
         DO 25 J=N3L1,N4
         F(I,J)=DSQRT (ST(K))*XI(I-NL2,J-N3)
25       CONTINUE
26       CONTINUE

*********************************
*        THE ELEMENTS [F34]         *
*********************************
         DO 28 I=N2L3,N3
         DO 27 J=N3L1,N4
         F(I,J)=2.0D0*DSQRT(TAY)*D(I-N2L2,J-N3)
27       CONTINUE
28       CONTINUE
```

```
*********************************
*       THE ELEMENTS [ F41]           *
*********************************
        DO 30 I=N3L3,N4
        DO 29 J=1,N
        F(I,J)=(-1.0D0)*XI(I-N3L2,J)
29      CONTINUE
30      CONTINUE
*******************************************
*       WE WILL USE NAGROUTINE (F02BJF)    *
*******************************************
        MATZ=.FALSE.
        EPS1 =(10.0D0)**(-15)
        IFAIL=0

        CALL F02BJF (N4,F,L,E,L,EPS1,ALFR,ALFI,BETA,MATZ,Z,L,ITER,IFAIL)
        J=0
        DO 31 I=1,N4
        IF (BETA(I).NE.0.0D0) THEN
        J=1+J
        ELANDAR(J)=ALFR(I)/BETA(I)
        ELANDAI(J)=ALFI(I)/BETA(I)
        END IF
31      CONTINUE

*       CHOOSING THE LARGEST REAL PART OF THE EIGENVALUE

        ELARGER=ELANDAR(1)
        DO 32 I=2,J
        IF (ELANDAR(I).GT.ELARGER) THEN
        ELARGER=ELANDAR(I)
        ELARGEI=ELANDAI(I)
        END IF
32      CONTINUE
        GR(K)=ELARGER
        GI(K)=ELARGEI
        IF (K.LT.2) GO TO 33
        IF (DABS(ST(K)-ST(K-1)).LT.1.0D-7)THEN
        R=ST(K)
        SEGMAR=GR(K)
        SEGMAI=GI(K)
        RETURN
        END IF
33      CONTINUE
```

```
      END

************************************************************************
*THIS SUBROUTINE CALCULATES THE PRODUCT OF TWO MATRICES      *
************************************************************************
      SUBROUTINE PRO(N,H,T,Z)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (L1=30)
      DIMENSION H(L1,L1),T(L1,L1),Z(L1,L1)
      DO 36,I=1,N
      DO 35 ,J=1,N
      Z(I,J)=0.0D0
      DO 34 K=1,N
      Z(I,J)=Z(I,J)+H(I,K)*T(K,J)
34    CONTINUE
35    CONTINUE
36    CONTINUE
      RETURN
      END
********************************************************************
*     THIS SUBROUTINE BULDS THE DERIVATIVE MATRIX                *
********************************************************************
      SUBROUTINE DRIVE (N,D)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (L1=30)
      DIMENSION D(L1,L1)
      DO 37 I=1,N
      DO 37 J=1,N
      IF ((J.GT.I).AND.(MOD(J-I,2).EQ.1))THEN
      D(I,J)=2.0D0* DBLE(J-1)
      ELSE
      D(I,J)=0.0D0
      END IF
37    CONTINUE
      DO 38 I=1,N
      D(1,I)=0.5D0*D(1,I)
38    CONTINUE
      RETURN
      END
```

Appendix 4

APPENDIX 4

```
**********************************************************************
*PROGRAMME OF BENARD PROBLEM WITH ROTATION AND RIGID                 *
*BOUNDARY                                                            *
*____________________________________________________________________*
*THERMAL CONVECTION IN A HORIZONTAL POROUS LAYER HEATED              *
*FROM BELOW AND AFFECTED BY ROTATION ABOUT THE THVERTICAL*
* AXIS. THE BOUNDARY CONDITIONS ARE RIGID(W=THETA=XI=0,DW=0)*
*AT Z=0,Z=1. IN THIS PROGRAMME APPROXIMATION OF CHEBYSHEV           *
*POLYNOMIALS IS USED TO SOLVE THE PROBLEM.                           *
*____________________________________________________________________*
      IMPLICIT DOUBLE PRECISION  (A-H,O-Z)
      PARAMETER (E=1.0D-9)
      COMMON N,R1,R2,PRV,PON,TAY
      OPEN (8,FILE='C:\WORK\THETA\RESULTrigid')

*     WHERE: (N) IS THE NUMBER OF POLYNOMIALS
*      (A & B) ARE TOW VALUES FOR THE WAVE NUMBER
*      (R1 & R2) ARE TOW VALUES FOR THE RAYLEIGH NUMBER
*      (PRV) THE VISCOUS PRANDTLE NUMBER
*      (PON) IS THE INVERSE OF THE DIMENSIONLESS PERMEABILITY
*      (TAY) IS THE TAYLOR NUMBER
*     (A1 & A2) AER TWO PARTICULAR POINTS

      WRITE(8,*)'INTER N,A,B,PRV,PON,TAY'
      READ*,N,A,B,R1,R2,PRV,PON,TAY
**********************************
*     RGOLD SECTION SEARCH      *
**********************************
      RGOLD =(DSQRT(5.0D0)-1.0D0)/2.0D0
      M=DABS(DLOG(E/(B-A))/(DLOG(RGOLD)))
      A1=A+RGOLD*(B-A)
      CALL CHEB (A1,U,G1R,G1I)
      A2= A+ RGOLD**2*(B-A)
      CALL CHEB (A2,V,G2R,G2I)
      DO 1 I=1,M
      IF (U.GT.V)THEN
      U=V
      B=A1
      A1=A2
      A2= A+ RGOLD**2*(B-A)
      CALL CHEB (A2,V,G2R,G2I)
```

```
      ELSE
      V=U
      A=A2
      A2=A1
      A1=A+RGOLD*(B-A)
      CALL CHEB (A1,U,G1R,G1I)
      END IF

      IF (DABS(U-V) .LE.E) GO TO 50
1     CONTINUE
50     AMIN=(A1 +A2)/2.0D0
      CALL CHEB (AMIN,R,SEGMAR,SEGMAI)
      WRITE(8,*) 'THE CRITICAL VALUE OF THE WAVE NUMBER =',AMIN
      WRITE(8,*) 'THE THERMAL RAYEILGH NUMBER =',R
      WRITE(8,*) 'THE REAL PART OF EIGENVALUE=',SEGMAR
      WRITE(8,*) 'THE IMAGINARY PART OF EIGEVALUE=',SEGMAI
      STOP
      END
*********************************************************************
*THIS SUBROUTINE IS USED TO EVALUATE THE RAYLEIGH NUMBER   *
*********************************************************************
      SUBROUTINE CHEB (A1,R,SEGMAR,SEGMAI)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON N,R1,R2,PRV,PON,TAY
      PARAMETER  (L1=50,L=4*L1)
      DIMENSION  F(L,L), E(L,L), XI(L1,L1), D(L1,L1), D2(L1,L1), V(L1,L1)
       DIMENSION  VPN(L1,L1), Z(L,L), ITER(L), ALFR(L), ALFI(L) ,ST(100)
       DIMENSION  BETA(L), GR(L), GI(L), ELANDAR(L), ELANDAI(L)
      LOGICAL  MATZ
      EXTERNAL F02BJF

      ST(1)= R1
      ST(2)= R2
      NL1=N+1
      NL2=N+2
      NL3=N+3
      N2=N*2
      N2L1=N2+1
      N2L2=N2+2
      N2L3=N2+3
      N3=N*3
      N3L1=N3+1
      N3L2=N3+2
      N3L3=N3+3
```

```
      N4=N*4
*************************************
*     THE IDENITY MATREX  (XI)      *
*************************************
      DO 3 I=1,N
      DO 2 J=1,N
      IF (I.EQ.J) THEN
       XI(I,J) = 1.0D0
      ELSE
       XI(I,J) =0.0D0
      END IF
2     CONTINUE
3     CONTINUE

      CALL DRIVE (N,D)
      CALL PRO (N,D,D,D2)

      DO 5 I=1,N
      DO 4 J=1,N
4     V(I,J)= 4.0D0*(D2(I,J)-(A1/2.0D0)**2*XI(I,J))
5     CONTINUE

      DO 7 I=1,N
      DO 6 J=1,N
6     VPN(I,J)= V(I,J)-PON*XI(I,J)
7     CONTINUE
**********************************
*     THE SECANT METHOD          *
**********************************
       DO 33 K=1,100
      IF (K.GE.3) THEN
      P=(ST(K-2)*GR(K-1)- ST(K-1)*GR(K-2))/(GR(K-1)- GR(K-2))
      ST(K)=dabs(P)
      END IF
*************************************
*     THE F & E ZEROING MATRICES     *
*************************************
      DO 9 I=1,N4
      DO 8 J=1,N4
      F(I,J)=0.0D0
      E(I,J)=0.0D0
8     CONTINUE
9     CONTINUE
*********************************
```

```
*    THE ELEMENTS [F11 & E11]       *
**********************************
        DO 11 I=3,N
        DO 10 J=1,N
        F(I,J)=VPN(I-2,J)
        E(I,J)=XI(I-2,J)
10      CONTINUE
11      CONTINUE
*************************************
*       THE ELEMENTS [F22 & E22]        *
*************************************
        DO 13 I=NL3,N2
        DO 12 J=NL1,N2
        F(NL1,J)=1.0D0
        F(NL2,J)=(-1.0D0)**(J-NL1)
        F(I,J)=V(I-NL2,J-N)
        E(I,J)=PRV*XI(I-NL2,J-N)
12      CONTINUE
13      CONTINUE
********************************
*       THE ELEMENTS [ E33]      *
********************************
        DO 15 I=N2L3,N3
        DO 14 J=N2L1,N3
        E(I,J)=XI(I-N2L2,J-N2)
14      CONTINUE
15      CONTINUE
*****************************
*       THE ELEMENTS [ F33]    *
*****************************
        DO 18 I=N2L3,N3
        DO 17 J=N2L1,N3
        F(N2L1,J)=1.0D0
        F(N2L2,J)=(-1.0D0)**(J-N2L1)
        F(I,J)= VPN(I-N2L2,J-N2)
17      CONTINUE
18      CONTINUE
********************************
*       THE ELEMENTS [ F44]         *
********************************
        DO 20 I=N3L3,N4
        DO 19 J=N3L1,N4
        F(N3L1,J)=1.0D0
        F(N3L2,J)=(-1.0D0)**(J-NL1)
```

```
      F(I,J)=V(I-N3L2,J-N3)
19    CONTINUE
20    CONTINUE
*********************************
*     THE ELEMENTS [ F12]       *
*********************************
      DO 22 I=3,N
      DO 21 J=NL1,N2
      F(I,J)=(-1.0D0)*DSQRT(ST(K))*(A1**2)*XI(I-2,J-N)
21    CONTINUE
22    CONTINUE
**********************************
*     THE ELEMENTS [ F13]        *
**********************************
      DO 24 I=3,N
      DO 23 J=N2L1,N3
      F(I,J)=(-2.0D0)*DSQRT(TAY)*D(I-2,J-N2)
23    CONTINUE
24    CONTINUE
**********************************
*     THE ELEMENTS [ F24]        *
**********************************
      DO 26 I=NL3,N2
      DO 25 J=N3L1,N4
      F(I,J)=DSQRT (ST(K))*XI(I-NL2,J-N3)
25    CONTINUE
26    CONTINUE
*********************************
*     THE ELEMENTS [F34]        *
*********************************
      DO 28 I=N2L3,N3
      DO 27 J=N3L1,N4
      F(I,J)=2.0D0*DSQRT(TAY)*D(I-N2L2,J-N3)
27    CONTINUE
28    CONTINUE
**********************************
*     THE ELEMENTS [ F41]        *
**********************************
      DO 30 I=N3L3,N4
      DO 29 J=1,N
      F(I,J)=(-1.0D0)*XI(I-N3L2,J)
29    CONTINUE
30    CONTINUE
*********************************
```

```
*       THE ELEMENTS [ F14]             *
*****************************
        DO 60 J=1,N
        F(1,J+N3)=2.0D0* DBLE((J-1)**2)
        F(2,J+N3)=2.0D0* DBLE((J-1)**2)*(-1.0D0)**(J)
60      CONTINUE
*****************************************
*       WE WILL USE NAGROUTINE (F02BJF)        *
*****************************************
        MATZ=.FALSE.
        EPS1 =(10.0D0)**(-15)
        IFAIL=0

        CALL F02BJF (N4,F,L,E,L,EPS1,ALFR,ALFI,BETA,MATZ,Z,L,ITER,IFAIL)
        J=0
        DO 31 I=1,N4
        IF (BETA(I).NE.0.0D0) THEN
        J=1+J
        ELANDAR(J)=ALFR(I)/BETA(I)
        ELANDAI(J)=ALFI(I)/BETA(I)
        END IF
31      CONTINUE

*       CHOOSING THE LARGEST REAL PART OF THE EIGENVALUE

        ELARGER=ELANDAR(1)
        DO 32 I=2,J
        IF (ELANDAR(I).GT.ELARGER) THEN
        ELARGER=ELANDAR(I)
        ELARGEI=ELANDAI(I)
        END IF
32      CONTINUE
        GR(K)=ELARGER
        GI(K)=ELARGEI
        IF (K.LT.2) GO TO 33
        IF (DABS(ST(K)-ST(K-1)).LT.1.0D-7)THEN
        R=ST(K)
        SEGMAR=GR(K)
        SEGMAI=GI(K)
        RETURN
        END IF
33      CONTINUE
        END
************************************************************
```

```
* THIS SUBROUTINE CALCULATES THE PRODUCT OF TWO MATRICES  *
****************************************************************
      SUBROUTINE PRO(N,H,T,Z)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (L1=50)
      DIMENSION H(L1,L1),T(L1,L1),Z(L1,L1)
      DO 36,I=1,N
      DO 35 ,J=1,N
      Z(I,J)=0.0D0
      DO 34 K=1,N
      Z(I,J)=Z(I,J)+H(I,K)*T(K,J)
34    CONTINUE
35    CONTINUE
36    CONTINUE
      RETURN
      END
*********************************************************
*     THIS SUBROUTINE BULDS THE DERIVATIVE MATRIX      *
*********************************************************
      SUBROUTINE DRIVE (N,D)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      PARAMETER (L1=50)
      DIMENSION D(L1,L1)
      DO 37 I=1,N
      DO 37 J=1,N
      IF ((J.GT.I).AND.(MOD(J-I,2).EQ.1))THEN
      D(I,J)=2.0D0* DBLE(J-1)
      ELSE
      D(I,J)=0.0D0
      END IF
37    CONTINUE
      DO 38 I=1,N
      D(1,I)=0.5D0*D(1,I)
38    CONTINUE
      RETURN
      END
```

المملكة العربية السعودية
وزارة التعليم العالي
جامعة أم القرى
كلية العلوم
قسم الرياضيات

الحمل الحراري لطبقة مسامية أفقية تتعرض لتأثير الدوران

بحث تكميلي ضمن متطلبات الحصول على درجة الماجستير في العلوم

تخصص الرياضيات التطبيقية

إعداد الطالبة

عبير بنت حبيب الله غلام بخش

بإشراف

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الدكتورة / ابتسام محمد أبو سليمان

١٤٣١هـ ــ ٢٠١٠ م

بسم الله الرحمن الرحيم

الإهداء

إلى تلك النجوم المتألقة في سمائي

إلى تلك الدرر الساكنة بأعماقي،،

إلى أمي الغالية وسرّ سعادتي،،

إلى والدي الغالي وسرّ عطائي،،

إلى أخي الكبير وأخواتي الرقيقات وسرّ أفراحي،،

إلى كل صديقة تألق بريق حبها في فضائي،،

إلى ذلك القلب الذي عرف سرّ أملي،،

،،،،

شكر وتقدير

إن كانت كلمات الشكر تعبر عن إحساسي..

فستطول سطور همساتي..

لكن هي معاني الوفاء تصل بحروف تتناسق مع مشاعر امتناني..

فأحمد الله تعالى أن يسر لي إنهاء بحثي..

ثم أشكر أهلي وسندي في دربي ..

وأشكر أستاذي الفاضل الدكتور عبد الله أحمد فهو معلمي..

وأشكر الدكتورة ابتسام ابو سليمان لمساعدتها لي..

وأشكر الدكتورة أمل أحمد العيدروس والدكتور خالد عبد العظيم مشالي لتفضلهم بمناقشة بحثي..

وأشكر رفيقتي دربي الأستاذة نفيسة الهندي والمعلمة سلطانه الأحمري..

وأشكر كل صديقة ومعلمة ساعدتني في مشوار بحثي ..

وأشكر كل أخٍ و أختٍ سعوا جاهدين لتسهيل عقباتي وبث الأمل إلى قلبي..

وأشكر كل قلب نبض معبرا عن حبه لي ..

فلترسم البسمةُ لكم معاني الشكر والوفاء على صفحاتي وسطوري..

وسأظل أدعو لكم بخير الجزاء من ربي العزيز الغفورِ..

وأن يجعل كل ذلك في موازين حسناتكم على مدى الدهورِ..

ملخص البحث

الحمل الحراري لطبقة مسامية أفقيه تتعرض لتأثير الدوران

هذا البحث يدرس الحمل الحراري لمسألة بينارد لطبقة مسامية أفقية لانهائية يتخللها مائع لزج وغير قابل للانضغاط وتؤثر عليها قوى التدوير . وتم مناقشة هذه المسألة بافتراض أن الطبقة المسامية تخضع لنموذج برينكمان كما تم مناقشة حالتي عدم الاستقرار الثابت والمفرط. تم دراسة المسألة من الناحية التحليلية في حالة الحدود الحرة، . وتم استنتاج أنه عندما يتم تسخين المائع من الأعلى، لا يمكن أن يحدث أي من حالات عدم الاستقرار، وعندما يتم تسخين المائع من الأسفل تحصل حالتي عدم الاستقرار الثابت والمفرط. كما تم إيجاد النتائج العددية في حالة الحدود الحرة و الحدود الصلبة وذلك باستخدام طريقة تشيبيشف. وتمت مناقشة تأثير الدوران و نفاذية الوسط المسامي على المسألة.و يحتوي البحث على ثلاثة فصول بالإضافة إلى المراجع والملحقات.

الفصل الأول: المقدمة.

الفصل الثاني: طريقة تشيبيشف لحل المسائل الحدية.

الفصل الثالث: الحمل الحراري لطبقة مسامية أفقية تتعرض لتأثير الدوران.